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Analysis of a domain decomposition method for linear transport problems on networks

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ABSTRACT

In this paper we analyze the convergence of the domain decomposition method applied to transport problems on networks. In particular, we derive estimates for the number of required iterations for linear problems. These estimates can be used to determine when the implementation of domain decomposition methods would be beneficial for this type of problems.

1. Introduction

Many physical processes can be, in first approximation, modeled as transport problems on networks; vehicular traffic flows [9,10], blood flow [6,15–17] or gas flow through pipe networks [2,5] are but three examples. Large networks can have thousands or even millions of edges, as in vascular systems for instance. The choice of efficient numerical strategies is thus of paramount importance. The goal of the paper is to analyze under what circumstances the use of domain decomposition (DD) methods is advantageous.

For simplicity, we consider *linear* transport problems on networks

$$\frac{\partial}{\partial t}\mathbf{q}_{e_i} + A_{e_i}\frac{\partial}{\partial x}\mathbf{q}_{e_i} = 0, \quad i = 1, \dots, M, \tag{1}$$

where e_i is the i-th edge of an M-edge network and where $\mathbf{q}_{e_i} \in \mathbb{R}^N$ stands for the state variables on e_i, $A_{e_i} \in \mathbb{R}^{N\times N}$ being a constant matrix associated to e_i. By assumption, our problem is one of transport; in other words, it is hyperbolic: each matrix A_{e_i} is diagonalizable with real eigenvalues. Junction conditions, usually algebraic constraints, have to be prescribed at the nodes of the network. These conditions can be quite complicated depending on the eigenstructure of A_{e_i}. To simplify our problem and focus on the analysis of the domain decomposition method, we consider the linear acoustics equations, which has symmetric eigenvalues and therefore relatively simple junction conditions can be imposed. While the details and numerical examples are given for these equations only, we show how the theory can be generalized to a linear constant coefficient system of the form (1), as long as appropriate junction conditions are given, which is very complicated in general, see Section 4.

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DD methods were originally developed for steady state problems. In this work, the spatial dimension is decomposed into the various edges of the network. When domain decomposition is applied to time dependent problems, at least two approaches are possible. First, DD can be applied spatially at each time step, see for instance [1,18]. Second, it can be applied in a spatio-temporal way, i.e., on cylinders corresponding to products of spatial subdomains with the entire computational time interval. This approach was introduced in [3,4] and later applied to the one-dimensional wave equation [7,8]. As detailed in Section 2, the second approach is used in this work, where the domains are the edges of the network.

Considerable work has been done in analyzing DD methods for transport problems defined on networks, see [12–14]. In this work, we use characteristics analysis to estimate the number of iterations necessary for the DD method to converge, disregarding numerical error. We show that this estimate depends upon on propagation speeds along each edge as well as on the length T of the considered time interval during which the problem is solved. This estimate is a key point in the cost study of the DD method for the present network problems. This analysis is carried out in Section 3. Finally, Section 5 compares numerical results to the theoretical estimates.

2. Linear acoustics equations on a network domain

Consider a network $\mathcal{G} = (V, E)$, where V is the set of vertices (nodes) of the network, and E is the set of edges. Let M denote the total number of edges in the network. Each edge e_i is modeled by an interval $[a_{e_i}, b_{e_i}]$, $a_{e_i} < b_{e_i}$, possibly with either $a_{e_i} = -\infty$ or $b_{e_i} = \infty$. A semi-infinite interval implies that the edge is an outlier, i.e., it is connected to the network on only one side.

We consider the particular transport problem given by the acoustics equations. We focus on this problem because of its simplicity, especially where junction conditions are concerned. The symmetric nature of the problem allows for easily implemented junction conditions and simplifies the convergence analysis of Section 3.

The linear acoustics equation on an arbitrary edge $e_i \in E$ of the network $\mathcal{G}$ reads

$$\left(\frac{\partial}{\partial t} + \begin{bmatrix} 0 & K_{e_i} \\ 1/\rho_{e_i} & 0 \end{bmatrix} \frac{\partial}{\partial x}\right) \begin{bmatrix} p_{e_i} \\ u_{e_i} \end{bmatrix}(x,t) = 0, \quad x \in (a_{e_i}, b_{e_i}), t > 0, \tag{2}$$

where p_{e_i} is the pressure and u_{e_i} is the velocity. The physical parameters $K_{e_i}, \rho_{e_i} > 0$ are assumed constant on the domain (a_{e_i}, b_{e_i}). Initial conditions are given by

$$p_{e_i}(x,0) = \phi_{e_i}(x), \qquad x \in [a_{e_i}, b_{e_i}], \tag{3}$$

$$u_{e_i}(x,0) = \psi_{e_i}(x), \qquad x \in [a_{e_i}, b_{e_i}]. \tag{4}$$

We assume $\phi_{e_i}(x) \in L^2([a_{e_i}, b_{e_i}])$, $\psi_{e_i}(x) \in L^2([a_{e_i}, b_{e_i}])$. Boundary conditions are introduced below when we consider the junction conditions for linear acoustics.

The eigenvalues of the matrix $\begin{bmatrix} 0 & K_{e_i} \\ 1/\rho_{e_i} & 0 \end{bmatrix}$ from (2) are given by

$$\lambda_1 = -c_{e_i} \qquad\qquad \lambda_2 = c_{e_i}$$

where $c_{e_i} = \sqrt{K_{e_i}/\rho_{e_i}} > 0$ is the speed of sound. The problem is thus strictly hyperbolic. Through a change of variables,

$$w_{e_i} = \frac{1}{2Z_{e_i}}(-p_{e_i} + Z_{e_i}u_{e_i}) \qquad z_{e_i} = \frac{1}{2Z_{e_i}}(p_{e_i} + Z_{e_i}u_{e_i})$$

where $Z_{e_i} = \rho_{e_i} c_{e_i}$, we can rewrite (2) as

$$\frac{\partial}{\partial t}\begin{bmatrix} w_{e_i} \\ z_{e_i} \end{bmatrix}(x,t) + \begin{bmatrix} -c_{e_i} & 0 \\ 0 & c_{e_i} \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} w_{e_i} \\ z_{e_i} \end{bmatrix}(x,t) = 0, \tag{5}$$

which is a decoupled system in terms of the characteristic variables w_{e_i}, and z_{e_i}. This form of the equations is used in the analysis below because of its direct connection to the speed and direction of propagation.

Given an edge e_i parameterized by the interval $[a_{e_i}, b_{e_i}]$, we denote the vertex corresponding a_{e_i} by e_i^a and the vertex corresponding to b_{e_i} by e_i^b. The various adjacent edges to e_i are also characterized in terms of their own orientation. More precisely, we set

$$\mathcal{A}_{e_i} = \{e_j \in E : e_j^b = e_i^a\}, \quad \mathcal{B}_{e_i} = \{e_j \in E : e_j^a = e_i^a\},$$

$$\mathcal{C}_{e_i} = \{e_j \in E : e_j^b = e_i^b\}, \quad \mathcal{D}_{e_i} = \{e_j \in E : e_j^a = e_i^b\}.$$

These adjacent edge sets are represented graphically in Fig. 1.

Since, for the above system, one eigenvalue is always positive and one is always negative, the resolution of the problem on edge e_i requires one boundary condition at each end. Alternatively, and equivalently, if we consider a vertex $v \in V$ along with $\mathcal{G}|_v$, the set of edges connected to v,