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Error analysis of a diffuse interface method for elliptic problems with Dirichlet boundary conditions

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ABSTRACT

We use a diffuse interface method for solving Poisson's equation with a Dirichlet condition on an embedded curved interface. The resulting diffuse interface problem is identified as a standard Dirichlet problem on approximating regular domains. We estimate the errors introduced by these domain perturbations, and prove convergence and convergence rates in the H^1-norm, the L^2-norm and the L^∞-norm in terms of the width of the diffuse layer. For an efficient numerical solution we consider the finite element method for which another domain perturbation is introduced. These perturbed domains are polygonal and non-convex in general. We prove convergence and convergences rates in the H^1-norm and the L^2-norm in terms of the layer width and the mesh size. In particular, for the L^2-norm estimates we present a problem adapted duality technique, which crucially makes use of the error estimates derived for the regularly perturbed domains. Our results are illustrated by numerical experiments, which also show that the derived estimates are sharp.

1. Introduction and main results

This paper considers the approximate solution of the following model problem by a diffuse interface method: Find $u \in H^1_0(\Omega)$ such that

$$-\Delta u = f \quad \text{in } \Omega \setminus \Gamma, \qquad u_{|\Gamma} = g_{|\Gamma} \quad \text{on } \Gamma. \tag{1}$$

Here, $f \in L^2(\Omega)$ models volume sources in a convex polygonal bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, and $g \in H^2(\Omega)$ defines the values of u on an interface $\Gamma \in C^{1,1}$, where $\Gamma \subset \Omega$ is a closed manifold of co-dimension one, i.e. for $n = 2$ a curve, or a surface if $n = 3$. We assume that the interface Γ separates Ω into two domains $\Omega = D_1 \cup \Gamma \cup D_2$, where $\Gamma = \partial D_1$, see Fig. 1.

The analysis of (1) is well-established. For instance, if g does not depend on u, (1) can be separated into two independent Dirichlet problems on D_1 and D_2 respectively, and the theory for the Poisson equation with Dirichlet boundary conditions applies, cf. [26,29]. Alternatively, one may formulate (1) as a Dirichlet problem on Ω constrained by $u = g$ on Γ and $u = 0$ on $\partial\Omega$, which leads to a saddle-point formulation, see e.g. [16,27].

The numerical approximation of (1) has been investigated intensively. A first set of numerical algorithms relies on a triangulation of Ω which is sufficiently aligned with the interface, i.e. Γ is approximated by a polygon, the segments of which are edges (or faces) of the elements; see e.g. [7,12–14,21,44]. The construction of such a triangulation might be expensive or difficult in practice. Furthermore, if $\Gamma = \Gamma(t)$ depends on time, problems similar to (1) need to be solved in each time step, and the triangulation has to be updated accordingly. In addition, one has to interpolate the data on

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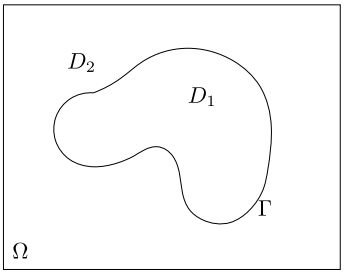

Fig. 1. Sketch of the geometry. $\Omega = D_1 \cup \Gamma \cup D_2$.

the varying meshes in this case. Therefore, a lot of research has been conducted to construct accurate methods employing meshes which are not aligned with Γ but possess a "simple" structure and are fixed throughout the simulation; see for instance the immersed boundary method [42], the immersed interface method [33,37], immersed finite elements [36,39,45], the fictitious domain method [5,27,28,40], the unfitted finite element method [7,19,31,34], the finite cell method [41], unfitted discontinuous Galerkin methods [9], or composite finite elements [30,38], and the references provided there.

In this work we will focus on a diffuse interface method for solving (1), see for instance [1,17,23,24,32,35,43]. In this method the sharp interface condition $u = g$ on Γ is replaced by suitable conditions on $u - g$ on a diffuse layer centered around Γ. Similar techniques have also been applied for solving coupled bulk-surface differential equations [1] or surface differential equations [10,20].

The constraint $u = g$ on Γ is equivalent to the condition $\|u - g\|_{L^2(\Gamma)} = 0$. In order to relax this condition on the sharp interface, we define the signed distance function

$$d_\Gamma(x) = \begin{cases} -\operatorname{dist}(x,\Gamma), & x \in D_1, \\ +\operatorname{dist}(x,\Gamma), & x \in D_2, \end{cases}$$

and we let $S : \mathbb{R} \to \mathbb{R}$ be such that $S(t) = t$ for $|t| < 1$ and $S(t) = \operatorname{sign}(t)$ for $|t| \geq 1$. Introducing a positive parameter ε, we define a regularized indicator function of D_1 as follows

$$\chi_{D_1}(x) \approx \omega^\varepsilon(x) = \frac{1}{2}\Big(1 + S\big(-\frac{d_\Gamma(x)}{\varepsilon}\big)\Big), \quad x \in \Omega.$$

Formally $\mathrm{d}\Gamma = |\nabla \chi_D|\,\mathrm{d}x \approx |\nabla \omega^\varepsilon|\,\mathrm{d}x$. Using the ε-tubular neighborhood $\mathcal{S}^\varepsilon = \operatorname{supp}(|\nabla \omega^\varepsilon|)$ of Γ, this leads to the following approximation for integrals along the interface

$$\|u - g\|^2_{L^2(\Gamma)} \approx \frac{1}{2\varepsilon} \int_{\mathcal{S}^\varepsilon} |u - g|^2 \,\mathrm{d}x.$$

The reader might find a more detailed derivation of this approximation and other choices of S in [17]. We further notice that, to make this approximation well-defined, we need the function g to be defined on the diffuse layer. If g is defined on Γ only, one has to use a suitable extension; for instance a local extension is given by $\tilde{g}(x + d_\Gamma(x)\nabla d_\Gamma(x)) = g(x)$ for $x \in \Gamma$, i.e. $\tilde{g}$ is constant off the interface. Thus, replacing the sharp interface constraint by the diffuse interface constraint $\int_{\mathcal{S}^\varepsilon} |u - g|^2 \,\mathrm{d}x = 0$, which amounts to $u = g$ on $\mathcal{S}^\varepsilon$, we are concerned with the following Dirichlet problem: Find $u^\varepsilon \in H^1_0(\Omega)$ such that

$$-\Delta u^\varepsilon = f \quad \text{in } \Omega \setminus \mathcal{S}^\varepsilon, \quad u^\varepsilon = g \quad \text{in } \mathcal{S}^\varepsilon. \tag{2}$$

Note that the particular choice of ω^ε is not important as long as $\mathcal{S}^\varepsilon = \operatorname{supp}(|\nabla \omega^\varepsilon|)$ is a ε-tubular neighborhood of Γ, i.e. methods using double obstacle potentials to regularize the indicator function of χ_D will essentially lead to the same method.

The main purpose of this paper is to estimate the errors introduced by this diffuse interface method; (i) on the continuous level and (ii) in a finite dimensional setting when using the finite element method, see below. The first result in this direction is the following approximation result on the continuous level.

Theorem 1.1. *Let $f \in L^2(\Omega)$ and $g \in H^2(\Omega)$, and let u be the solution to (1), and let u^ε be the solution to (2). Then there exists a constant $C > 0$ independent of ε such that*

$$\frac{1}{\varepsilon}\|u - u^\varepsilon\|_{L^2(\Omega)} + \frac{1}{\sqrt{\varepsilon}}\|\nabla u - \nabla u^\varepsilon\|_{L^2(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)}\big).$$

The numerical approximation of (2) is still not straight-forward as the (sufficiently exact) integration over $\mathcal{S}^\varepsilon$ is basically not easier than the integration along Γ. In order to obtain an efficient numerical scheme let $\mathcal{T}_h$ be a shape regular triangulation of Ω, where $h = \max\{\operatorname{diam}(T) : T \in \mathcal{T}_h\}$ denotes the mesh size parameter, and let