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Newton-type methods for inverse singular value problems with multiple singular values

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ABSTRACT

We consider the convergence problem of some Newton-type methods for solving the inverse singular value problem with multiple and positive singular values. Under the nonsingularity assumption of the relative generalized Jacobian matrices at the solution $\mathbf{c}^*$, a convergence analysis for the multiple and positive case is provided and the superlinear or quadratical convergence properties are proved. Moreover, numerical experiments are given in the last section and comparisons are made.

1. Introduction

Inverse singular value problems (ISVPs) have a growing importance in practical applications such as the optimal sequence designed for direct-spread code division multiple access [29], the passivity enforcement in nonlinear circuit simulation [25], the constructions of Toeplitz-related matrices from prescribed singular values [2], the inverse problem in some quadratic group [19], and others [17,18,23,32]. For different kinds of ISVPs, one may refer to [8,31].

In this paper, we consider a special kind of ISVP defined as follows. Let $\{\sigma_i^*\}_{i=1}^n$ be given with $\sigma_1^* \geq \sigma_2^* \geq \cdots \geq \sigma_n^* \geq 0$. Let $\{A_i\}_{i=0}^n$ be a sequence of m-by-n matrices with $m \geq n$. Given $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$, we define

$$A(\mathbf{c}) := A_0 + \sum_{i=1}^{n} c_i A_i \tag{1.1}$$

and denote the singular values of $A(\mathbf{c})$ by $\{\sigma_i(\mathbf{c})\}_{i=1}^n$ with the order $\sigma_1(\mathbf{c}) \geq \sigma_2(\mathbf{c}) \geq \cdots \geq \sigma_n(\mathbf{c}) \geq 0$. Then the ISVP considered here is to find a vector $\mathbf{c}^* \in \mathbb{R}^n$ such that

$$\sigma_i(\mathbf{c}^*) = \sigma_i^*, \quad \text{for each } i = 1, 2, \ldots, n. \tag{1.2}$$

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The vector $\mathbf{c}^*$ is called a solution of the ISVP (1.2).

This kind of ISVP was originally proposed by Chu [6] in 1992. For more discussion on this ISVP, one may refer to [7,8]. For symmetric matrices, the ISVP (1.2) is essentially the same as the inverse eigenvalue problem (IEP) studied extensively in [1,5,7,8,13,26,31]. Even though the solvability issue for the ISVP is very complicated, some numerical algorithms for solving (1.2) have still been developed [3,4,6,15,30]. In general, these numerical methods can be distinguished into two classes. One is the continuous method which consists of solving an ordinary differential obtained from an explicit calculation of the projected gradient of a certain objective function (cf. [6]). The other kind of method that we are interested in below is the iterative methods. Notice that solving the ISVP (1.2) is equivalent to solving the equation $\mathbf{f}(\mathbf{c}) = \mathbf{0}$ in $\mathbb{R}^n$, where the function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\mathbf{f}(\mathbf{c}) := (\sigma_1(\mathbf{c}) - \sigma_1^*,\ \sigma_2(\mathbf{c}) - \sigma_2^*,\ \ldots,\ \sigma_n(\mathbf{c}) - \sigma_n^*)^T, \quad \text{for any } \mathbf{c} \in \mathbb{R}^n. \tag{1.3}$$

In the case when the given singular values are distinct and positive, i.e.,

$$\sigma_1^* > \sigma_2^* > \cdots > \sigma_n^* > 0,$$

the function $\mathbf{f}$ is differentiable around the solution $\mathbf{c}^*$, and the singular vectors corresponding to $\{\sigma_i(\mathbf{c})\}_{i=1}^n$ are continuous with respect to $\mathbf{c}$ around $\mathbf{c}^*$ (cf. [4]). Based on the equivalence of the ISVP (1.2) and the nonlinear equation $\mathbf{f}(\mathbf{c}) = \mathbf{0}$, Newton's method for solving nonlinear equations can certainly be applied to solving the ISVP (1.2). However, Newton's method requires solving a complete singular value problem for the matrix $A(\mathbf{c})$ at each outer iteration. This sometimes makes it inefficient from the viewpoint of practical calculations especially when the problem size is large. By using the Cayley transform, a Newton-type method was designed in [6] for the ISVP (1.2) which generalized an effective iterative process proposed originally by Friedland, Nocedal, and Overton [13] for solving the IEP. Under the assumption that the given singular values are distinct and positive, the Newton-type method was proved in [3] to converge quadratically in the root sense. On the other hand, in order to alleviate the over-solving problem [10], Bai et al. proposed in [4] an inexact Newton-type method where a suitable stopping criterion was chosen for the approximate Jacobian equation. Also under the assumption that the given singular values are distinct and positive, a convergence analysis for the inexact Newton-type method was provided and the superlinear convergence property was proved [4]. Recently, Vong, Bai, and Jin presented in [30] an Ulm-like method for the ISVP which avoids solving approximate Jacobian equations. Again under the assumption that the given singular values are distinct and positive, they showed that the proposed method converged at least quadratically.

However, in the case when multiple or zero singular values are present, that is, without loss of generality, $\{\sigma_i^*\}_{i=1}^n$ satisfies that

$$\sigma_1^* = \cdots = \sigma_t^* > \sigma_{t+1}^* \cdots > \sigma_n^* \geq 0, \tag{1.4}$$

solving the ISVP (1.2) becomes much more complicated and the technique used for convergence analysis of the algorithms mentioned above for the positive and distinct case would not work again. Indeed, when multiple or zero singular values are present, the differentiability of $\mathbf{f}$ and the continuity of the singular vectors corresponding to the multiple singular values (which play crucial roles in establishing the convergence results for these algorithms; see [3,4]) fail in the case of multiple or zero singular values. Following a strategy of Friedland et al. in [13], Chu presented in [6] a different kind of ISVP rather than (1.2) under assumption (1.4) with $\sigma_n^* > 0$, and proposed a modified Newton-type method for solving it. While in paper [15], a regularized directional derivative-based Newton method (together with its convergence analysis) was proposed for solving the ISVP (1.2) with multiple and positive singular values which, at each iteration, involves both computation of a directional derivative for a semismooth function (not smooth in general) and solving of nonlinear (not linear in general) equations. However, to our knowledge, Newton-type methods similar to the ones of the distinct and positive case for solving the ISVP (1.2) with multiple or zero singular values have not been explored. Particularly, Vong et al. mentioned in [30] that extending their method and convergence analysis to the cases of multiple singular values and of zero singular values is an interesting topic.

The purpose of this paper is to modify the Newton-type methods (including the exact and inexact versions) proposed in [4,6] for the distinct and positive case to suit for the multiple and positive case, and to study their convergence issue. We propose a new nonsingularity assumption to establish the superlinear convergence of the inexact Newton-type method; this nonsingularity assumption is in terms of the relative generalized Jacobian matrices evaluated at the solution $\mathbf{c}^*$ and is motivated by the corresponding ones in [26,28] for the IEP with multiple eigenvalues. In particular, as a corollary of the convergence theorem of the inexact version, a quadratic convergence result of the (exact) Newton-type method is obtained. It should be noted that the techniques used here for the convergence analysis are different from the ones for the distinct and positive case because of the absence of the differentiability of $\mathbf{f}$ and the continuity of the singular vectors as we explained above. Numerical experiments are presented to illustrate our results, and comparisons between the inexact Newton-type method and the (exact) Newton-type method are made.

The paper is organized as follows. In section 2, we give some preliminaries. In section 3, we present the Newton-type methods for solving the ISVP (1.2) with multiple singular values, and establish the convergence results of the proposed methods. Numerical tests are reported in section 4.