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Weak convergence for a stochastic exponential integrator and finite element discretization of stochastic partial differential equation with multiplicative & additive noise

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ABSTRACT

We consider a finite element approximation of a general semi-linear stochastic partial differential equation (SPDE) driven by space-time multiplicative and additive noise. We examine the full weak convergence rate of the exponential Euler scheme when the linear operator is self adjoint and also provide the full weak convergence rate for non-self-adjoint linear operator with additive noise. Key part of the proof does not rely on Malliavin calculus. For non-self-adjoint operators, we analyse the optimal strong error for spatially semi-discrete approximations for both multiplicative and additive noise with truncated and non-truncated noise. Depending on the regularity of the noise and the initial solution, we found that in some cases the rate of weak convergence is twice the rate of the strong convergence. Our convergence rate is in agreement with some numerical results in two dimensions.

1. Introduction

The weak numerical approximation of an Itô stochastic partial differential equation defined in the bounded domain $\Lambda \subset \mathbb{R}^d$ is analyzed. Boundary conditions on the domain Λ are typically Neumann, Dirichlet or Robin conditions. More precisely, we consider in the abstract setting the following stochastic partial differential equation

$$dX = (AX + F(X))dt + B(X)dW, \qquad X(0) = X_0, \qquad t \in [0, T], \qquad T > 0 \tag{1}$$

on the Hilbert space $L^2(\Lambda)$ with more emphasis on additive noise. Here the linear operator A which is not necessarily self adjoint, is the generator of an analytic semigroup $S(t) := e^{tA}, t \geq 0$. The functions F and B are nonlinear functions of X and the noise term $W(t)$ is a Q-Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{F_t\}_{t\geq 0})$, that is white in time. The filtration is assumed to fulfill the usual conditions (see e.g. [36, Definition 2.1.11]). For technical reasons more interest will be on a deterministic initial value $X_0 \in H$. The noise can be represented as a series in the eigenfunctions of the covariance operator Q given by

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$$W(x,t) = \sum_{i\in\mathbb{N}^d} \sqrt{q_i} e_i(x)\beta_i(t), \tag{2}$$

where (q_i, e_i), $i \in \mathbb{N}^d$ are the eigenvalues and eigenfunctions of the covariance operator Q and β_i are independent and identically distributed standard Brownian motions. Under some technical assumptions it is well known (see [8,36,7]) that the unique mild solution of (1) is given by

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dW(s). \tag{3}$$

Equations of type (1) arise in physics, biology and engineering [39,40,15] and only in few cases, exact solutions are explicitly available. The study of numerical solutions of SPDEs is therefore an active research area and there is an extensive literature on numerical methods for SPDEs of the form (1) [19–21,40,32,1,30,46]. Basically there are two types of convergence. The strong convergence or pathwise convergence studies the pathwise convergence of the numerical solution to true solution while the weak convergence aims to approximate the law of the solution at a fixed time. In many applications, weak error is more relevant as interest is usually in a function of the solution i.e., $\mathbb{E}\Phi(X)$, where $\Phi : H \to \mathbb{R}$ and $\mathbb{E}$ is the expectation. Strong convergence rates for numerical approximations of stochastic evolution equations of type (1) with smooth and regular nonlinearities are well understood in the scientific literature (see [19–21,1,30,46,40,33] and references therein). Weak convergence rates for numerical approximations of equation (1) are far away from being well understood. For a linear SPDE with additive noise, the solution can be written explicitly and the weak error have been estimated in [38,26,10] with implicit Euler method for time discretization. The space discretization has been performed with finite difference method [38,26] and finite element method [26]. The weak error of the implicit Euler method is more complicated for nonlinear equation of type (1) as the Malliavin calculus is usually used to handle the irregular term and the term involving the nonlinear operators F and B (see [9,45,2]). Weak error in space for nonlinear equations has been studied in [5] for spectral Galerkin method and in [2] for finite element method. The work in [5] does not rely on Malliavin calculus. In almost all the literature for weak error estimation, the linear operator A is assumed to be self adjoint. Furthermore no numerical simulations are made to sustain the theoretical results and only standard Dirichlet boundary are used to the best of our knowledge. In this paper we consider a stochastic exponential scheme (called stochastic exponential Euler scheme) as in [33] and provide the weak error of the full discrete scheme (Theorem 6.2, Proposition 6.1 and Remark 6.4) where the space discretization is performed using finite element following closely the works in [44,45] on another exponential integrator scheme. Key part of our weak convergence proof does not use Malliavin calculus. Furthermore for additive noise, our weak convergence proof covers the case where the linear operator A is not necessarily self-adjoint with optimal order in time and complex boundary condition rather than the standard Dirichlet boundary condition mostly used in the literature. Recent work in [2] is used to obtain optimal convergence order both in time and space for additive noise when the linear operator is self adjoint in Proposition 6.1 and Remark 6.4, and some numerical examples to sustain the theoretical results are provided. We also extend in Theorem 6.1 the strong optimal convergence rate provided in [27, Theorem 1.1] to non-self adjoint operator A with truncated and non-truncated noise. Note that as the operator A is not necessarily self-adjoint, our scheme here is based on exponential matrix computation. The deterministic part of this scheme have been proven to be efficient and robust in comparison to standard schemes in many applications [15,40–42] where the exponential matrix functions have been computed using the Krylov subspace technique [18] and fast Leja points technique [4]. For convenience of presentation, we take A to be a second order operator as this simplifies the convergence proof. Our results can be extended to high order semi-linear parabolic SPDE.

The paper is organized as follows. Section 2 provides the abstract setting and the well posedness of (1). The stochastic exponential Euler scheme along with weak error representation is provided in Section 3. The temporal weak convergence rate of the stochastic exponential Euler scheme is provided in Section 4 for additive noise and in Section 5 for multiplicative noise. Note that in this section the linear operator A is not necessarily self-adjoint. Section 6 provides strong optimal convergence rate of the semi-discrete solution for non-self-adjoint operator A along with full weak convergence rate of the stochastic exponential Euler scheme. For multiplicative noise, we assume that $A = \Delta$, $Q = I$ and $d = 1$ with homogeneous Dirichlet boundary condition. Numerical results to sustain some theoretical results are provided in Section 7.

2. The abstract setting and mild solution

Let us start by presenting briefly the notation for the main function spaces and norms that we use in the paper. Let H be a separable Hilbert space with the norm $\|\cdot\|$ associated to the inner product $\langle,\rangle_H$. For a Hilbert space U we denote by $\|\cdot\|_U$ the norm of U, $L(U,H)$ the set of bounded linear mapping from U to H and by $L^2(\Omega,U)$[1] the Hilbert space of all equivalence classes of square integrable U-valued random variables. For ease of notation $L(U,U) = L(U)$. Furthermore we

[1] Ω is the sample space.