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Fast iterative solvers for large matrix systems arising from time-dependent Stokes control problems

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ABSTRACT

In this manuscript we consider the development of fast iterative solvers for Stokes control problems, an important class of PDE-constrained optimization problems. In particular we wish to develop effective preconditioners for the matrix systems arising from finite element discretizations of time-dependent variants of such problems. To do this we consider a suitable rearrangement of the matrix systems, and exploit the saddle point structure of many of the relevant sub-matrices involved – we may then use this to construct representations of these sub-matrices based on good approximations of their $(1, 1)$-block and Schur complement. We test our recommended iterative methods on a distributed control problem with Dirichlet boundary conditions, and on a time-periodic problem.

1. Introduction

Of late, the efficient numerical solution of PDE-constrained optimization problems has been an active area of research (see [30] for an introduction to PDE-constrained optimization). One of the main classes of such problems is that of Stokes control problems (which arise from flow control), and this is the class that we seek to examine in this paper. Many researchers have sought to develop solution strategies for the matrix systems resulting from finite element discretizations of time-independent [20,29,32] and time-dependent [7,10,11,28] versions of these setups. It is a substantial challenge in particular to build solvers for the complex matrix systems that arise in the time-dependent case, and this is the problem on which we focus in this paper.

When discretized using finite elements, the resulting matrix systems are of very high dimension, even in comparison to equivalent time-independent formulations, and are also sparse. It is therefore extremely desirable to develop fast and robust iterative methods for their solution. We do this by exploiting the saddle point structure of the relevant matrices to construct effective preconditioners for the entire system. Previous research into this problem has also exploited the structure of the matrix systems: in [7] multigrid routines were developed using appropriate smoothers and prolongation/restriction operators, and in [28] saddle point preconditioners which exhibited mesh-independence were constructed and applied within MINRES. In this paper a preconditioned MINRES method is also proposed, but with the goal that the solver exhibits favorable convergence properties as both mesh-size and regularization parameter are modified.

In this paper we consider such problems with different types of conditions at initial (and final) time: both initial conditions and time-periodic conditions. We wish our solvers for these problems to be fast and effective for a range of parameter values, to involve the storage of small matrices compared with the dimension of the entire system, and to be parallelizable.

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To do this we build on work undertaken by the author in [20,21] when solving matrix systems from time-independent Stokes and Navier–Stokes control problems, to tackle the larger and more complex systems arising when a time-dependent component is introduced. We find that this methodology can reasonably be applied to the time-dependent set-up, and we wish to present numerical results that validate this assertion.

This paper is structured as follows. In this section we state the problems that we wish to examine, and introduce some basic theory of saddle point systems. In Section 2 we derive the matrix systems of which the solution is required; in Section 3 we devise preconditioning strategies for these systems, which may be applied within a suitable iterative method. In Section 4 we present numerical results to demonstrate the performance of our methods, and finally in Section 5 we make some concluding comments.

1.1. Problem statement

The two problems on which we wish to focus in this paper are both time-dependent variants of the widely-considered Stokes control problem. The first is the following distributed control problem with Dirichlet boundary conditions:

$$
\begin{aligned}
(\mathbf{P1}) \qquad & \min_{\vec{v},\vec{u}} \ \frac{1}{2}\int_0^T\int_\Omega \left\|\vec{v}-\vec{v}_d\right\|^2 \ \mathrm{d}\Omega\mathrm{d}t + \frac{\beta}{2}\int_0^T\int_\Omega \left\|\vec{u}\right\|^2 \ \mathrm{d}\Omega\mathrm{d}t \\
\text{s.t.} \quad & \frac{\partial\vec{v}}{\partial t} - \nabla^2\vec{v} + \nabla p = \vec{u}, && \text{in } \Omega\times[0,T], \\
& -\nabla\cdot\vec{v} = 0, && \text{in } \Omega\times[0,T], \\
& \vec{v}(\mathbf{x},t) = \vec{f}(\mathbf{x},t), && \text{on } \partial\Omega\times[0,T], \\
& \vec{v}(\mathbf{x},0) = \vec{g}(\mathbf{x}), && \text{on } \Omega.
\end{aligned}
$$

This problem is solved for spatial coordinates given by $\mathbf{x}\in\mathbb{R}^d$, $d\in\{2,3\}$, and time t, on the space-time domain $\Omega\times[0,T]$ with boundary $\partial\Omega\times[0,T]$. The *state variables* for this problem are given by the velocity $\vec{v}$ (defined in d dimensions) and the pressure p. The *control variable* is denoted as $\vec{u}$ (in d dimensions), and $\vec{v}_d$ defines the *desired state*. The positive parameter β denotes the *regularization parameter* (or *Tikhonov parameter*), and indicates at what ratio one prioritizes the realization of a state variable close to the desired state as opposed to the minimization of the control. The functions $\vec{f}$ and $\vec{g}$ are defined over the spatial coordinates (and in the case of $\vec{f}$ over time as well), and correspond to boundary conditions and initial conditions respectively. We note that it would be equally feasible to include a natural boundary condition of the form $\frac{\partial\vec{v}(\mathbf{x},t)}{\partial\vec{n}} - p\vec{n} = \vec{f}(\mathbf{x},t)$ (where $\vec{n}$ is the outward facing normal vector to Ω, and $\frac{\partial}{\partial\vec{n}}$ denotes the normal derivative), or indeed a Neumann/mixed boundary condition, instead of a Dirichlet condition. The methodology introduced in this paper could also be tailored to these problems; however the performance of our preconditioner may change or degrade slightly, as has been observed for the forward problem.

The second problem is of a similar flavor to that stated above, but is a time-periodic problem. We write this as

$$
\begin{aligned}
(\mathbf{P2}) \qquad & \min_{\vec{v},\vec{u}} \ \frac{1}{2}\int_0^T\int_\Omega \left\|\vec{v}-\vec{v}_d\right\|^2 \ \mathrm{d}\Omega\mathrm{d}t + \frac{\beta}{2}\int_0^T\int_\Omega \left\|\vec{u}\right\|^2 \ \mathrm{d}\Omega\mathrm{d}t \\
\text{s.t.} \quad & \frac{\partial\vec{v}}{\partial t} - \nabla^2\vec{v} + \nabla p = \vec{u}, && \text{in } \Omega\times[0,T], \\
& -\nabla\cdot\vec{v} = 0, && \text{in } \Omega\times[0,T], \\
& \vec{v}(\mathbf{x},t) = \vec{f}(\mathbf{x},t), && \text{on } \partial\Omega\times[0,T], \\
& \vec{v}(\mathbf{x},0) = \vec{v}(\mathbf{x},T), && \text{on } \Omega.
\end{aligned}
$$

Here there is no longer an initial condition corresponding to a given function, but instead a restriction that the velocity profile must be the same at initial and final times. This creates different features when attempting to solve the problem at hand.

The problems, as well as the matrix systems arising from them, are of complex structure, and so finding ways to solve these problems efficiently is a highly non-trivial task. In the next section, we discuss the form of matrices which we may often consider when carrying out the solution process.

1.2. Saddle point systems

When discretizing the problems **(P1)** and **(P2)** using a finite element method, the resulting matrix systems are of *saddle point* form. We therefore wish to briefly introduce such systems and some widely used approximations of them.