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Primal hybrid method for parabolic problems

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ABSTRACT

In this article, a class of second order parabolic initial-boundary value problems in the framework of primal hybrid principle is discussed. The interelement continuity requirement for standard finite element method has been alleviated by using primal hybrid method. Finite elements are constructed and used in spatial direction, and backward Euler scheme is used in temporal direction for solving fully discrete scheme. Optimal order estimates for both the semidiscrete and fully discrete method are derived with the help of modified projection operator. Numerical results are obtained in order to verify the theoretical analysis.

1. Introduction

Consider a second order parabolic model problem

$$u_t(x,t) - \Delta u(x,t) = f(x,t) \quad \text{in } \Omega \times (0,T], \tag{1.1}$$

$$u(x,t) = 0 \quad \text{on } \Gamma \times (0,T], \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{1.3}$$

where Ω is an open, bounded convex polyhedral subset of $\mathbb{R}^n$ with Lipschitz continuous boundary Γ, T is the fixed final time, $u_t = \frac{\partial u}{\partial t}$, Δ denotes $\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, f and u_0 are appropriate smooth functions.

The weak formulation of problem (1.1)–(1.3) is to find $u : [0,T] \to H_0^1(\Omega)$ such that

$$(u_t, v) + (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega),\ t > 0,$$

$$u(0) = u_0,$$

where $(\cdot,\cdot)$ is the usual inner product in $L^2(\Omega)$. Standard finite element methods for solving (1.1)–(1.3) are based on the above formulation and are largely studied, we refer to Thomée [19], Wheeler [20]. In these methods a finite dimensional space of $H_0^1(\Omega)$ is constructed made up with functions which are continuous along interelement boundaries.

The interelement continuity condition for the finite dimensional space which approximate $H_0^1(\Omega)$ is a strong condition and can be relaxed which is well known as nonconforming methods. In nonconforming methods the finite dimensional

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space is no longer a subspace of $H^1_0(\Omega)$. For a detailed analysis of these methods for solving second order elliptic equations we refer to Crouzeix and Raviart [8], Iron and Razzaque [9] and Strang [16].

A more general approach known as primal hybrid method in which the interelement continuity is withdrawn by introducing a Lagrange multiplier. This type of method is firstly introduced by engineers Pian and Tong [13] and is viewed as a generalization of nonconforming methods. The primal hybrid method for second order elliptic problems is proposed by Raviart and Thomas [15] which is based on an extended variational principle. Similarly, the dual hybrid method for second order elliptic problems is proposed by Thomas [18] which is based on a complementary energy principle. Primal hybrid method for fourth order elliptic problem is discussed by Quarteroni [14]. A primal hybrid method for a strongly nonlinear second order elliptic problem is discussed by Park [11]. Primal hybrid methods for second order quasi-linear elliptic problems is studied by Milner [10]. An application of the primal hybrid method to an optimal shape problem is established in [5]. The idea of primal hybrid method is used to develop some nonconforming domain decomposition methods, for instance see [2,12] for mortar finite element method with Lagrange multipliers. There are a few literatures available on the primal hybrid method for elliptic problems, but hardly any articles are available on primal hybrid method applied to parabolic problems. For instance, error estimates using cell discretization method for some parabolic problems are discussed by Swann [17] which is a generalization of primal hybrid method. Therefore, it is important to study these methods further. For a general discussion on hybrid methods we refer to [4].

In this article, we propose a finite element approximations of the parabolic initial-boundary value problems (1.1)–(1.3) in the primal hybrid context. Optimal order estimates for semidiscrete methods for both primal and hybrid variables are established. Using backward Euler method, a fully discrete scheme is derived and optimal order error estimates are developed. The error analysis are performed with the help of a modified elliptic projection and an orthogonal projection. Numerical results are obtained to validate the theoretical estimates.

A brief outline of this article is as follows. In Section 2, we introduce some functional spaces required for our analysis purpose. In Section 3, we discuss some approximation spaces to derive primal hybrid formulation for the original problem and then we establish error analysis for the semidiscrete scheme. Section 4 is devoted to fully discrete scheme with error analysis. We give some numerical results in Section 5. Finally, we conclude in Section 6.

2. Preliminaries

We define the Sobolev spaces which are used in the sequel. For a non-negative integer m, we define $H^m(0,T;Y)$ as

$$H^m(0,T;Y) = \left\{ v : (0,T) \to Y : \sum_{j=0}^{m} \int_0^T \left\| \frac{\partial^j v}{\partial t^j} \right\|_Y^2 dt < \infty \right\} \tag{2.1}$$

and is equipped with the norm

$$||v||_{H^m(0,T;Y)} = \left(\sum_{j=0}^{m} \int_0^T \left\| \frac{\partial^j v}{\partial t^j} \right\|_Y^2 dt \right)^{1/2},$$

where Y is a Banach space with a norm $||\cdot||_Y$. For $m = 0$, it corresponds to the space $L^2(0,T;Y)$. We define a multi index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ as a n-tuple of non-negative integers α_i, $1 \le i \le n$ with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and set

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

The Sobolev space of order m is defined as

$$H^m(\Omega) = \left\{ v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega), |\alpha| \le m \right\}$$

provided with a norm and a semi-norm

$$||v||_{m,\Omega} = \left(\sum_{|\alpha| \le m} \int_\Omega |D^\alpha v|^2 dx \right)^{1/2}, \quad |v|_{m,\Omega} = \left(\sum_{|\alpha| = m} \int_\Omega |D^\alpha v|^2 dx \right)^{1/2},$$

respectively. For $m = 0$, it corresponds to the space $L^2(\Omega)$ and further we denote the L^2-norm by $||\cdot||_{L^2(\Omega)}$. We define a negative norm $||\cdot||_{-m,\Omega}$ by:

$$||v||_{-m,\Omega} = \sup_{0 \ne \phi \in H^m(\Omega)} \frac{(v,\phi)}{||\phi||_{m,\Omega}},$$

where $v \in L^2(\Omega)$.