Applied Numerical Mathematics 108 (2016) 125–142

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

An inexact low-rank Newton–ADI method for large-scale algebraic Riccati equations

Peter Benner [a], Matthias Heinkenschloss [b,1], Jens Saak [a], Heiko K. Weichelt [c,*,2]

[a] Research Group Computational Methods in Systems and Control Theory (CSC), Max Planck Institute for Dynamics of Complex Technical Systems Magdeburg, Sandtorstr. 1, 39106 Magdeburg, Germany
[b] Department of Computational and Applied Mathematics (CAAM), Rice University, MS-134, 6100 Main Street, Houston, TX 77005-1892, USA
[c] The Mathworks Ltd., Matrix House, Cambridge Business Park, CB4 0HH Cambridge, United Kingdom

ARTICLE INFO

Article history:
Received 14 September 2015
Received in revised form 23 April 2016
Accepted 30 May 2016
Available online 4 June 2016

Keywords:
Riccati equation
Kleinman–Newton
Inexact solves
Low-rank ADI methods
Line search

ABSTRACT

This paper improves the inexact Kleinman–Newton method for solving algebraic Riccati equations by incorporating a line search and by systematically integrating the low-rank structure resulting from ADI methods for the approximate solution of the Lyapunov equation that needs to be solved to compute the Kleinman–Newton step. A convergence result is presented that tailors the convergence proof for general inexact Newton methods to the structure of Riccati equations and avoids positive semi-definiteness assumptions on the Lyapunov equation residual, which in general do not hold for low-rank approaches. In the convergence proof of this paper, the line search is needed to ensure that the Riccati residuals decrease monotonically in norm. In the numerical experiments, the line search can lead to substantial reduction in the overall number of ADI iterations and, therefore, overall computational cost.

1. Introduction

We present improvements of the inexact Kleinman–Newton method for the solution of large-scale continuous-time algebraic Riccati equations (CARE)

$$\mathcal{R}(X) = C^T C + A^T X + XA - XBB^T X = 0 \tag{1.1}$$

with $C \in \mathbb{R}^{p\times n}$, $A \in \mathbb{R}^{n\times n}$, $X = X^T \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, and $p + r \ll n$. The algorithmic improvements consist of incorporating a line search and of systematically integrating the low-rank structure resulting from the ADI method for the solution of the Lyapunov equation

$$(A^{(k)})^T X^{(k+1)} + X^{(k+1)} A^{(k)} = -C^T C - X^{(k)} BB^T X^{(k)}, \tag{1.2}$$

* Corresponding author.
E-mail addresses: benner@mpi-magdeburg.mpg.de (P. Benner), heinken@rice.edu (M. Heinkenschloss), saak@mpi-magdeburg.mpg.de (J. Saak), heiko.weichelt@mathworks.co.uk (H.K. Weichelt).

[1] The research of this author was supported in part by grants AFOSR FA9550-12-1-0155 and NSF DMS-1115345.
[2] This work was completed in large parts while this author was with the Max Planck Institute for Dynamics of Complex Technical Systems Magdeburg, Sandtorstr. 1, 39106 Magdeburg, Germany.

http://dx.doi.org/10.1016/j.apnum.2016.05.006
0168-9274/

where

$$A^{(k)} = A - BB^T X^{(k)},$$

which has to be approximately solved in the k-th iteration. The paper is motivated by the recent work of Feitzinger et al. [10] who propose and analyze an inexact Kleinman–Newton method without line search, by Benner and Byers [3] who incorporate line search into the exact Kleinman–Newton method, and by the recent work of Benner et al. [4,6] on algorithmic improvements of low-rank ADI methods. The convergence result in [10] makes positive semi-definiteness assumptions on the difference between certain matrices and the residual of the Lyapunov equation that are in general not valid when the Lyapunov equation is solved with low-rank methods like, e.g., the low-rank ADI iteration [8]. Our convergence result follows the theory of general inexact Newton methods, but uses the structure of Riccati equations. We add the inexact solution of the Lyapunov equation to [3] and incorporate the low-rank structure.

Our convergence proof makes use of the fact that the Riccati residuals decrease monotonically in norm, which is ensured by the line search. There is no proof that the inexact low-rank Kleinman–Newton–ADI iteration converges globally without line search. On test examples resulting from the finite element approximation of LQR problems governed by an advection diffusion equation, the incorporation of a line search into the inexact low-rank Kleinman–Newton–ADI iteration can lead to substantial reduction in the overall number of ADI iterations and, therefore, overall computational cost.

The paper is organized as follows. In the next section, we recall a basic existence and uniqueness result for the unique symmetric positive semi-definite stabilizing solution of the CARE (1.1). Section 3 introduces the inexact Kleinman–Newton method with line search and presents the basic convergence result. The basic ingredients of ADI methods that are needed for this paper are reviewed in Section 4. Section 5 discusses the efficient computation of various quantities like the Newton residual using the low-rank structure. As a result, the computational cost of our overall algorithm is proportional to the total number of ADI iterations used; in comparison the cost of other components, such as execution of the line search, are negligible. Finally, we demonstrate the contributions of the various improvements on the overall performance gains in Section 6. As mentioned before, in our numerical tests, our improved inexact Kleinman–Newton method is seven to twelve times faster than the exact Kleinman–Newton method without line search.

Notation. Throughout the paper we consider the Hilbert space of matrices in $\mathbb{R}^{n\times n}$ endowed with the inner product $\langle M, N\rangle = \operatorname{tr}\left(M^T N\right) = \sum_{i,j=1}^{n} M_{ij}N_{ij}$ and the corresponding (Frobenius) norm $\|M\|_F = (\langle M, M\rangle)^{1/2} = (\sum_{i,j=1}^{n} M_{ij}^2)^{1/2}$. Furthermore, given real symmetric matrices M, N, we write $M \succeq N$ if and only if $M - N$ is positive semi-definite, and $M \succ N$ if and only if $M - N$ is positive definite. The spectrum of a symmetric matrix M is denoted by $\sigma(M)$.

2. The Riccati equation

We recall an existence and uniqueness result for the continuous-time Riccati equation (1.1).

Definition 1. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, and $C \in \mathbb{R}^{p\times n}$. The pair (A, B) is called stabilizable if there exists a feedback matrix $K \in \mathbb{R}^{n\times r}$ such that $A - BK^T$ is stable, which means that $A - BK^T$ has only eigenvalues in the open left half complex plane $\mathbb{C}^-$. The pair (C, A) is called detectable if (A^T, C^T) is stabilizable.

Notice that (A, B) is stabilizable if and only if (A, BB^T) is stabilizable and (C, A) is detectable if and only if (C^TC, A) is detectable. Furthermore, we always use the word stable as defined in [16], whereas, in other literature, this is usually called *asymptotically stable*. Since, as in [16], asymptotically stable is the required property in all our applications we do not need to distinguish between stable and asymptotically stable and, therefore, simply use stable everywhere.

Assumption 2. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, and $C \in \mathbb{R}^{p\times n}$ are given such that (A, B) is stabilizable and (C, A) is detectable.

If Assumption 2 holds, there exists a unique symmetric positive semi-definite solution $X^{(*)}$ of the CARE (1.1) which is also the unique stabilizing solution. This follows from Theorems 8.5.1 and 9.1.2 (see also p. 244) in [16].

Furthermore, we show next that all symmetric positive semi-definite solutions of the CARE (1.1) are stabilizing. This result is important in our context, since we generate symmetric positive semidefinite iterates $X^{(k)}$ and require that $A^{(k)}$ is stable.

Theorem 3. *If Assumption 2 holds, every symmetric solution $X^{(*)} \succeq 0$ of the CARE (1.1) is stabilizing.*

Proof. Let $X = X^T \succeq 0$ solve the CARE (1.1). We show that $A - BB^TX$ is stable by contradiction.

Assume that μ is an eigenvalue of $A - BB^TX$ with $\operatorname{Re}(\mu) \geq 0$ and let $v \in \mathbb{C}^n\backslash\{0\}$ be a corresponding eigenvector. The CARE (1.1) can be written as

$$(A - BB^TX)^TX + X(A - BB^TX) = -C^TC - XBB^TX. \tag{2.1}$$