Applied Numerical Mathematics 108 (2016) 143–156

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Convolution regularization method for backward problems of linear parabolic equations

Cong Shi, Chen Wang, Ting Wei *

School of Mathematics and Statistics, Lanzhou University, Gansu 730000, PR China

ARTICLE INFO

Article history:
Received 27 March 2015
Received in revised form 14 September 2015
Accepted 30 December 2015
Available online 7 June 2016

Keywords:
Backward problems for parabolic equations
A priori parameter choice
A posteriori parameter choice
Convolution regularization method
Generalized discrepancy principle

ABSTRACT

In this paper, we consider a class of severely ill-posed backward problems for linear parabolic equations. We use a convolution regularization method to obtain a stable approximate initial data from the noisy final data. The convergence rates are obtained under an *a priori* and an *a posteriori* regularization parameter choice rule in which the *a posteriori* parameter choice is a new generalized discrepancy principle based on a modified version of Morozov's discrepancy principle. The log-type convergence order under the *a priori* regularization parameter choice rule and log log-type order under the *a posteriori* regularization parameter choice rule are obtained. Two numerical examples are tested to support our theoretical results.

1. Introduction

In this paper, we consider a class of backward problems for parabolic equations, given by

$$u_t(x,t) + Lu(x,t) = 0, \quad x \in \mathbb{R}, \quad t \in (0,T), \tag{1.1}$$

$$u(x,T) = f(x), \quad \lim_{|x|\to\infty} u(x,t) = 0, \tag{1.2}$$

where L is a differential operator with respect to x, such as the second order elliptic operator $Lu = -(u_{xx} + \nu u_x - q)$ with constant coefficients and the space-fractional diffusion operator $Lu = {}_xD_\theta^\alpha u$ defined in next section.

The backward problems for parabolic equations are very important in various practical applications and there have many researches on these problems [3,6–8,20,21,23,29,34,18,38]. We will see in Section 2 that the backward problem (1.1)–(1.2) is severely ill-posed, refer to [23,34]. Therefore, a suitable regularization method should be used (e.g. [37]). In this paper, we consider a class of convolution regularization methods proposed firstly in [39]. That is, to find an approximate solution for the backward problem (1.1)–(1.2) by solving a new well-posed problem

$$(u_\mu^\delta)_t + P_\mu(x) * Lu_\mu^\delta = 0, \tag{1.3}$$

$$u_\mu^\delta(x,T) = f^\delta(x), \quad \lim_{|x|\to\infty} u_\mu^\delta(x,t) = 0, \tag{1.4}$$

where "$*$" denotes the convolution operation, and the family of functions $P_\mu(x)$, which involves a regularization parameter μ, is a suitably-chosen convolution kernel and f^δ is a noisy function to f.

* Corresponding author.
E-mail address: tingwei@lzu.edu.cn (T. Wei).

http://dx.doi.org/10.1016/j.apnum.2015.12.009
0168-9274/

The convolution regularization is a new method, and there are a few papers currently. The convolution method is related to the mollification method in [30,14] but there is an essential difference. The convolution method aims at mollifying the equation but the mollification method aims at mollifying the improper data [30,14]. For example, Manselli, Miller [26] and Murio [31,32] have used mollification methods to solve some ill-posed problems for the heat equation, but their method is working only on the Weierstrass kernel. In [14], Hào gave a choice of the mollification parameter and obtained a convergence estimate for a non-characteristic Cauchy problem of parabolic equations.

The regularization parameter may be chosen by either an *a priori* or an *a posteriori* method. The *a priori* choice is pretty straightforward and have been studied extensively, but it has a drawback that the *a priori* bound for the exact solution is unknown, and an incorrect bound will lead to a bad approximation, and sometimes we will prefer an *a posteriori* method. In general, the error estimate under the *a posteriori* parameter choice is hard to obtain, but it is more suitable in practical applications. We note that the authors have not provided an *a posteriori* parameter choice method in [39]. In this paper, we mainly focus on this point.

The most widely used *a posteriori* parameter choice method is Morozov's discrepancy principle [10,12,17,33,35,36]. The various generalizations of Morozov's discrepancy principle are developed in [1,2,4,5,11,13,16,22,24,10,19]. In this paper, Morozov's discrepancy principle can not be used directly for obtaining a convergence rate. Thus we use a generalized discrepancy principle to choose the regularization parameter, which is different from Morozov's discrepancy principle and its variations in the references mentioned above. The corresponding error estimates between the exact solution and the regularized solution can be derived by a carefully analysis. Two numerical examples are provided to verify the effectiveness of our proposed methods.

The paper is organized as follows. In Section 2, the convolution regularization method and the choice of convolution kernels are discussed. In Sections 3 and 4, we derive the convergence rates under the *a priori* and *a posteriori* rules for the choices of the regularization parameter. In Section 5 we give two numerical examples to illustrate the effectiveness of the *a posteriori* and the *a priori* choice rules. Finally, we give a conclusion in Section 6.

2. Convolution regularization method

We assume throughout the paper that all the functions involving x belong to $L^2(\mathbb{R})$, and $\|\cdot\|$ always denotes the L^2 norm, i.e.

$$\|f\| = \left(\int_{\mathbb{R}} |f(x)|^2 dx \right)^{1/2} \tag{2.1}$$

and the function space $H^p(\mathbb{R})$ is defined by:

$$\left\{ f(x) | f \in L^2(\mathbb{R}), \|f\|_p := \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} (1+\omega^2)^p |\hat{f}(x)|^2 dx \right)^{1/2} < +\infty \right\}. \tag{2.2}$$

It is a very natural idea to analyze pseudo-differential operators or the convolution operation in the frequency domain space, so we need to use the Fourier transform. The Fourier transform is given by

$$\hat{f}(\omega) = \int_{\mathbb{R}} e^{i\omega x} f(x) dx, \tag{2.3}$$

and the Parseval formula holds

$$\sqrt{2\pi}\|f\| = \|\hat{f}\|. \tag{2.4}$$

We define the operator L as a (possibly fractional) differential operator with respect to x, mapping from $L^\infty((0,T); H^k(\mathbb{R}))$ to $L^\infty((0,T); H^{k-\alpha}(\mathbb{R}))$, given by the Fourier transform:

$$\widehat{Lu}(\omega,t) = l(\omega)\hat{u}(\omega,t), \tag{2.5}$$

and $l : \mathbb{R} \to \mathbb{C}$ is a function called the *symbol* of the operator L, satisfying the following conditions:

(L_1) $|l(\omega)| \le a(1+|\omega|^\alpha)$, $\omega \in \mathbb{R}$, for $a > 0$ and $\alpha > 0$;
(L_2) $\mathrm{Re}\, l(\omega) \ge a_1|\omega|^\alpha$, $\omega \in \mathbb{R}$, for $a_1 > 0$ and $\alpha > 0$.

The motivation of this definition is that some integer and fractional differential operators can be written as multipliers over the frequency space, and their growth as a function of ω is polynomial. The number $\alpha > 0$ indicates the order of the operator L. Here are some examples of well-known evolution equations written in the following forms