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# An inverse scattering problem with generalized oblique derivative boundary condition

Haibing Wang, Jijun Liu *

*Department of Mathematics, Southeast University, Nanjing 210096, PR China*

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ABSTRACT

Consider the scattering of long ocean tidal waves by an island taking into account the influence of daily rotation of the Earth, which is modeled by an exterior boundary value problem for the two-dimensional Helmholtz equation with generalized oblique derivative boundary condition. In this paper, we are concerned with a corresponding inverse scattering problem which is to reconstruct the unknown obstacle (island) from the far-field data. After proving the unique solvability of the direct scattering problem in a suitable function space required for our inverse scattering problem, we establish the linear sampling method (LSM) for reconstructing the boundary of the obstacle from the far-field data. To clarify the validity of such a sampling-type method which essentially depends on the solvability of an interior boundary value problem, we show that, except a discrete set of wave numbers, such an interior problem has a unique solution. Finally, some numerical examples are presented to demonstrate the efficiency of the reconstruction scheme.


## 1. Introduction

Consider the diffraction of tidal waves by an island on water of constant finite depth. Let $u^i$ be an incident tidal wave, and $u^s$ be the corresponding scattered wave by the island with cross section $D \subset \mathbb{R}^2$. If we take into account the influence of daily rotation of the Earth on the ocean waves, then under some assumptions the total wave $u := u^i + u^s$ satisfies

$$\begin{cases} \Delta u + k^2 u = 0, & x \in \mathbb{R}^2 \setminus \overline{D}, \\ \dfrac{\partial u}{\partial \nu} + i\lambda \dfrac{\partial u}{\partial \tau} = 0, & x \in \partial D, \\ \lim\limits_{r\to+\infty} \sqrt{r}\left(\dfrac{\partial u^s}{\partial r} - iku^s\right) = 0, & r = |x|, \end{cases} \tag{1.1}$$

where $\frac{\partial u}{\partial \nu}$ and $\frac{\partial u}{\partial \tau}$ are the normal and tangential derivatives of $u$ on $\partial D$, respectively. And also,

$$\lambda := f_0/\omega \tag{1.2}$$

is a real dimensionless parameter with $|\lambda| < 1$, and

$$k^2 := (\omega^2 - f_0^2)/(gh) > 0, \tag{1.3}$$

* Corresponding author.
*E-mail addresses:* hbwang@seu.edu.cn (H. Wang), jjliu@seu.edu.cn (J. Liu).

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where $\omega$ is the frequency of the time-harmonic motion, $f_0$ is the Coriolis parameter, $g$ is the acceleration due to gravity and $h$ is the depth of the ocean; see [17] for the derivation of this model.

The boundary condition

$$\frac{\partial u}{\partial \nu} + i\lambda \frac{\partial u}{\partial \tau} = 0 \quad \text{on } \partial D \tag{1.4}$$

is called the *generalized* oblique derivative boundary condition and differs from the Dirichlet, Neumann or impedance boundary condition due to the appearance of the tangential derivative $\frac{\partial u}{\partial \tau}$ with complex coefficient $i\lambda$. If the coefficient $i\lambda$ is replaced by a real one, we are led to the classical oblique derivative boundary condition in the sense that the combination of the normal and tangential derivatives can be written as a directional derivative in a certain direction. In the case of complex coefficient $i\lambda$, it was found recently in [28] that some fundamental results for wave scattering such as the symmetric property of the Green function and the reciprocity principle of the scattered waves should be modified. Therefore both the direct and inverse scattering problems for our model (1.1) are important and interesting mathematically.

In [17], Martin established the unique solvability of the direct scattering problem (1.1) using the integral equation method. By expressing the scattered field as a single-layer potential, the problem (1.1) is reformulated as a boundary integral equation which involves a Cauchy singular integral term. Recently, in [28], the authors proved the unique solvability of the problem (1.1) directly by a PDE-based method and gave the characterization of the Green function associated with the boundary value problem (1.1). Krutitskii considered in [13] the scattering of tidal waves by spits or reefs, where $D$ in (1.1) degenerates into an open curve. The well-posedness for this model is justified by using the integral equation method with the use of angular potential. There are also some works on the other oblique derivative problems, for example, see [12,19–21,29].

Compared with the researches on the direct scattering problem for (1.1), the corresponding inverse scattering problems, which aim to determine the boundary of island from some information about the scattered wave, are still in the initial stage. Although the uniqueness of reconstructing $\partial D$ from the far-field data was proven in [28] theoretically, the efficient reconstruction schemes are absent for this new model.

In this paper, we consider the problem of reconstructing $\partial D$ from the far-field patterns of incident plane waves for our scattering model (1.1). A reconstruction scheme called the linear sampling method (LSM) is investigated. As we know, except for some well-known works for inverse scattering problems, for example [2,3,7,9,22], LSM has been extensively studied and applied to many inverse scattering problems with classical boundary conditions; see [1,4–6] and the references therein. The further investigations from the numerical aspects for LSM are given in [14–16,26,27]. However, to the best of the authors' knowledge, there are no implementations of LSM for the inverse scattering models with the generalized oblique derivative boundary condition. Due to some new properties of the boundary value problem (1.1), the justification of LSM and its numerical implementation for our new model are not trivial due to the tangential derivative in the boundary condition. Theoretically, the linear sampling method is valid only when $k^2$ is not an eigenvalue of the corresponding interior boundary value problem. Hence, the existence of eigenvalues as well as their distributions for the corresponding interior problem plays a central role in our reconstruction scheme. We will show that the eigenvalues for the interior oblique derivative problem form at most a discrete set, and therefore LSM is still applicable to our inverse scattering problem for almost all wave numbers.

This paper is organized as follows. In Section 2, we show the unique solvability of the direct scattering problem using the integral equation method. Compared with the work in [17], we explicitly show this solvability by proving that the boundary integral equation for the density function is solvable in $H^{-1/2}(\partial D)$, which is required for our reconstruction scheme applied to the inverse scattering problem. Section 3 is devoted to the justification of LSM for our scattering model, where some new techniques are needed due to the modified symmetry property of the Green function for the oblique derivative problem (1.1). In Section 4, the distribution of the eigenvalues for the interior oblique derivative problem is studied, which ensures the applicability of LSM to our inverse scattering problem for almost all wave numbers $k$. In Section 5, some numerical examples are presented to show the performance of the reconstruction scheme in our setting.

## 2. Unique solvability of the direct scattering problem

In this section, we consider the following general boundary value problem

$$\begin{cases} \Delta u^s + k^2 u^s = 0, & x \in \mathbb{R}^2 \setminus \overline{D}, \\ \dfrac{\partial u^s}{\partial \nu} + i\lambda \dfrac{\partial u^s}{\partial \tau} = f, & x \in \partial D, \\ \lim\limits_{r\to+\infty} \sqrt{r}\left(\dfrac{\partial u^s}{\partial r} - iku^s\right) = 0, & r = |x| \end{cases} \tag{2.1}$$

for any $f \in H^{-1/2}(\partial D)$. We establish the unique solvability of the direct scattering problem in $H^1(\mathbb{R}^2 \setminus \overline{D})$ using the integral equation method. Then the solvability of (2.1) is applicable to our wave scattering problem with the special boundary data