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On the kernel and particle consistency in smoothed particle hydrodynamics

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ABSTRACT

The problem of consistency of smoothed particle hydrodynamics (SPH) has demanded considerable attention in the past few years due to the ever increasing number of applications of the method in many areas of science and engineering. A loss of consistency leads to an inevitable loss of approximation accuracy. In this paper, we revisit the issue of SPH kernel and particle consistency and demonstrate that SPH has a limiting second-order convergence rate. Numerical experiments with suitably chosen test functions validate this conclusion. In particular, we find that when using the root mean square error as a model evaluation statistics, well-known corrective SPH schemes, which were thought to converge to second, or even higher order, are actually first-order accurate, or at best close to second order. We also find that observing the joint limit when $N \to \infty$, $h \to 0$, and $n \to \infty$, as was recently proposed by Zhu et al., where N is the total number of particles, h is the smoothing length, and n is the number of neighbor particles, standard SPH restores full C^0 particle consistency for both the estimates of the function and its derivatives and becomes insensitive to particle disorder.

1. Introduction

The method of smoothed particle hydrodynamics (SPH) was introduced in the literature independently by Gingold and Monaghan [7] and Lucy [15] for modeling astrophysical flow problems. Since then the method has been widely applied to different areas of science and engineering due to its simplicity and ease of implementation. At the same time, the method has been improved over the years to overcome major shortcomings and deficiencies. One longstanding drawback of standard SPH is the particle inconsistency, which is an intrinsic manifestation of the lack of integrability of the kernel approximation in its spatially discretized form, resulting in a loss of accuracy. In practical applications, SPH inconsistency arises when the

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support domain of the kernel is truncated by a model boundary, for irregularly distributed particles even in the absence of kernel truncation, and for spatially adaptive calculations where a variable smoothing length is employed.

Several corrective strategies have been proposed to restore particle consistency in SPH calculations. A simple correction technique was first advanced by Li and Liu [11] and Liu et al. [13], where the kernel itself is modified to ensure that polynomial functions up to a given degree are exactly interpolated. A kernel gradient correction, allowing for the exact evaluation of the gradient of a linear function, was further proposed by Bonet and Lok [2] based on a variational formulation of SPH. A general approach to the construction of kernel functions that obey the consistency conditions of SPH in continuous form and describe the compact supportness requirement was presented by Liu et al. [14]. A drawback of this approach is that the reconstructed smoothing functions may be partially negative, non-symmetric, and non-monotonically decreasing, thereby compromising the stability of the numerical simulations.

More stable approaches for restoring consistency are based on Taylor series expansions of the kernel approximations of a function and its derivatives. In general, if up to m derivatives are retained in the series expansions, the resulting kernel and particle approximations will have $(m+1)$th-order accuracy or mth-order consistency (i.e., C^m consistency). This approach was first developed by Chen et al. [5,4] (their corrective smoothed particle method, or CSPM), which solves for the approximation of a function separately from that of its derivatives by neglecting all terms involving derivatives in the former expansion and retaining only first-order terms in the latter expansions. This scheme is equivalent to a Shepard's interpolation for the function [20], and so it should restore C^1 kernel and particle consistency for the interior regions and C^0 consistency at the boundaries. By retaining only first-order derivatives in the Taylor series expansions for the function and its derivatives and solving simultaneously the resulting set of linear equations (the FPM scheme), Liu and Liu [12] argued that C^1 kernel and particle consistency can be obtained for both interior and boundary regions. The FPM scheme was further improved by Zhang and Batra [26] (their MSPH scheme), where now up to second-order derivatives are retained in the Taylor series expansions. In principle, this method should restore C^2 consistency for the SPH approximation of the function (i.e., third-order convergence rates) and C^1 consistency for the first-order derivatives. However, when adding higher-order derivatives in the Taylor expansions, the number of algebraic linear equations to be solved increases rapidly, implying high computational costs. In addition, since the solution involves a matrix inversion, for some types of problems the stability of the scheme can be compromised by the conditioning of the matrix. A modified FPM approach, which is free of kernel gradients and leads to a symmetric corrective matrix was recently proposed by Huang et al. [10]. An alternative formulation based on the inclusion of boundary integrals in the kernel approximation of the spatial derivatives was reported by Macià et al. [16], which restores C^0 consistency at the model boundaries. A new SPH formulation, based on a novel piecewise high-order Moving-Least-Squares WENO reconstruction and on the use of Riemann solvers, that improves the accuracy of SPH in the presence of sharp discontinuities was recently reported by Avesani et al. [1].

A new strategy to ensure formal convergence and particle consistency with standard SPH has recently been devised by Zhu et al. [28] in the astrophysical context. In this approach, no corrections are required and full consistency is recovered provided that $N \to \infty$, $h \to 0$, and $n \to \infty$, where N is the total number of particles, h is the smoothing length, and n is the number of neighbor particles within the kernel support. They found that if n is fixed, as is customary in SPH, consistency will not be guaranteed even though $N \to \infty$ and $h \to 0$ since there is a residual error that will not vanish unless n is allowed to increase with N as $n \sim N^{1/2}$. However, the systematic increase of n with improved resolution demands changing the interpolation kernel to a Wendland-type function [25], which, unlike traditional kernels, is free from the pairing instability when used to perform smoothing in SPH with large numbers of neighbors [6].

In this paper, we revisit the issue of kernel and particle consistency in SPH. We first demonstrate that the normalization condition of the kernel is independent of h, suggesting that its discrete representation depends only on n, consistently with the error analyses of Vaughan et al. [23] and Read et al. [21]. Although C^0 and C^1 kernel and particle consistency can be achieved by some corrective SPH methods, a simple observation shows that C^2 kernel consistency is difficult to achieve, implying an upper limit to the convergence rate of SPH in practical applications. Numerical experiments with suitably chosen test functions in two-space dimensions validate this conclusion. The paper is organized as follows. In Section 2, we discuss the issue of C^0 consistency and show that the normalization condition of the kernel is independent of h. The issue of higher-order consistency is considered in Section 3, where we show that C^2 consistency is affected by an inherent intrinsic diffusion, which arises as a consequence of the dispersion of the SPH particle positions relative to the mean. Section 4 outlines the importance of restoring C^1 consistency for the gradient. Finally, Section 5 presents numerical tests that demonstrate the convergence rates of the particle approximations and Section 6 contains the conclusions.

2. Normalization condition and C^0 consistency

As it is well-known, the starting point of SPH lies on the exact identity

$$f(\mathbf{x}) = \int_{\mathcal{R}^3} f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')d\mathbf{x}', \tag{1}$$

where $f = f(\mathbf{x})$ is some sufficiently smooth function, $\delta(\mathbf{x}-\mathbf{x}')$ is the Dirac-δ distribution, and the integration is taken over all space. The kernel approximation is obtained by replacing the Dirac-δ distribution by some kernel interpolation function W such that