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On the relation of the Embedded Discontinuous Galerkin method to the stabilized residual-based finite element methods

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A R T I C L E I N F O

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A B S T R A C T

The paper traces the relation between the Embedded Discontinuous Galerkin method and the Streamline Upwind Petrov–Galerkin method and its generalizations, such as the Discontinuous Residual-free Bubble method, for the Euler and Navier–Stokes equations. Popular choices of the stabilization terms for both discretizations are related on the analytical level. The conservation property of the Embedded Discontinuous Galerkin discretization on dual volumes of the computational grid is established.

1. Introduction

A high level of expectation exists within the CFD community with regard to the development of high-order methods. It is envisaged that, once implemented within the production CFD solvers (steady and unsteady), they will dramatically reduce the time needed for CFD analysis, and with that to design cycles. The ultimate hope is that, combined with the adjoint-based techniques for *a posteriori* error estimation and anisotropic grid adaptation, high-order methods will allow to quantify and efficiently reduce below any practically reasonable tolerance the numerical errors usually significantly contributing to the CFD results for real-life applications.

As for the existing approaches to the design of high-order methods, finite element (FE) discretizations have quite a few widely recognized advantages over the finite volume (FV) and finite difference schemes. These include relative compactness of the stencils used in the approximation of the differential operators; the ability to universally and, in many cases, rigorously, treat a wide variety of the boundary conditions, and relative technical ease with which high-order schemes of basically any order can be implemented. In general, the FE approach for general unstructured grids is supported by much more developed mathematical apparatus than the FV/FD approaches: in particular, the problem of approximation is universally solved by appealing to the optimality property of the piecewise polynomial FE spaces in the Sobolev's functional spaces [9].

Although the variety of finite element discretizations devised to deal with the advection–diffusion systems, such as the steady-state Euler, Navier–Stokes and Reynolds–Averaged NS equations of the fluid dynamics, is extremely diverse, the approaches to stabilization for these schemes fall pretty well into the following two large domains. The first employs polynomial spaces with finite elements which are allowed to be discontinuous across the element interfaces. The reconciliation

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of the advective fluxes on these interfaces is done via the approximate Riemann solvers. The most representative family of approaches in this group is built around the Discontinuous Galerkin (DG) method, originally by Baker [2]. This approach turned out to be so general and flexible that in later decades it gave rise to an impressive amount of extensions and variations. The DG discretization uses the same mechanism for stabilization in the advection limit as the FV methods. This natural way of stabilization in DG has, however, to be combined with a rather non-trivial treatment of the diffusion terms with discontinuous polynomial spaces, a challenge DG methods share with the mixed finite element methods for the elliptic equations.

The second stabilization approach starts from the finite element spaces of the globally continuous functions for representing the solution of the PDE. The stabilization is achieved by modifying the space of the test functions within the Petrov–Galerkin approach. The representative discretizations in this class are the Streamline Upwind Petrov–Galerkin (SUPG) scheme, originally formulated by Hughes and Brooks [7] and its variants, such as, Galerkin–Least Squares (GLS) [18]. These schemes can be characterized as *residual* schemes, as the stabilization they use is proportional to the strong residual of the underlying PDE: the property naturally resulting in the consistency of these types of discretizations. It was not immediately realized that in the core of this second approach to stabilization lies the universal deep principle of variational multiscale resolution [17,15,20]: the paradigm reducing the problem of stability of the FE discretization to the question of accuracy of the unresolved scales of the solution as represented on a given grid. This realization allowed to relate the SUPG/GLS discretizations to such approaches as Residual Free Bubbles (RFB) and several others, originally devised and developed independently [6].

Impressive progress has been achieved in the last 2 decades in the unification of various FE schemes: within the DG- and SUPG-type approaches to stabilization, respectively For the SUPG-type schemes this was done within the aforementioned multiscale paradigm. For the DG-type approaches the influential paper [1] unifies and classifies a wide variety of the approaches to the stabilization with discontinuous FE spaces in the diffusion limit, and, in particular, includes the types of stabilizations which were not originally devised as DG schemes (e.g., certain interface penalization approaches). For the more novel, Hybridizable Discontinuous Galerkin (HDG) schemes (the ones of utmost interest for us in this study) Cockburn, Gopalakrishnan and Lazarov in [10] have built an unification framework, which, in particular, allowed them to devise new, hybridizable variants of the classical DG schemes.

Every substantial unification attempt so far has contributed to not just a better understanding of the mechanisms behind the stabilization techniques for the advection-dominated PDEs, but also to the improvement of existing schemes and to the construction of new ones. On the other hand, the connection of the two alternative concepts to stabilization is currently understood only on quite abstract, conceptual level. See, for example, the paper of Brezzi et al. [5] which discusses, within another general paradigm — the least-squares approach to stabilization, the common form of stabilization terms appearing in both Discontinuous Galerkin and residual-based methodologies.

As noted above, the actual diversity in the stabilized FE schemes for the advection–diffusion systems is very high. A few FE schemes exist which combine the ideas of using discontinuous FE spaces to achieve the DG-type stabilization with the use of a globally continuous representation of the solution. Three such schemes are the Embedded Discontinuous Galerkin scheme (EDG), which is the primal object for the current study, the Multiscale Discontinuous Galerkin (MDG) scheme by Hughes et al. [19] and the Discontinuous Residual Free Bubble (DRFB) method by G. Sangalli [24]. The MDG and the DRFB methods fit well into the paradigm of the variational multiscale-resolution approach. They are of interest for us in this study because of their relation to EDG. It is also possible to combine the use of discontinuous finite element spaces with the Petrov–Galerkin approach as does the DPG method by Demkowicz and Gopalakrishnan [12], though in a different way.

In this paper we relate the stabilization of certain DG-type schemes, namely, EDG, to the stabilization used in SUPG and multiscale approaches. This connection is traced on not just a conceptual level (though this aspect is also important for this study) but also on an analytical level. In particular, we show how to derive, under certain essential assumptions, the SUPG discretization for both the Euler and Navier–Stokes equations, from the basically Riemann solver-based stabilization employed by the HDG and EDG methods. This allows one to get insight into the analytical structure of both stabilization mechanisms and propose possible improvements, working both ways. Along the way, the present analysis establishes the conservation property of the EDG scheme over the dual volumes of the simplicial, unstructured grid, and provides some insights into the structure of the flux-based boundary conditions for the Euler equations when discretized with the EDG/HDG approaches. In the case of the advection systems the proposed analysis of the EDG discretization allows to accurately track the differences between the EDG and MDG discretizations and to basically prove the equivalence of the EDG discretization to the DRFB approach.

The layout of the paper is as follows. We start in section 2 by considering the steady state Euler equations. This section is auxiliary in the sense that we explain the formalism for dealing with fluxes, test and trial functions and such, which is then used throughout the presentation. In section 3 and the beginning of section 4 we review very briefly how the HDG and EDG discretization are constructed for the Euler equations. The rest of the section 4 deals with the analysis of the EDG scheme. We recast this scheme in a form similar in structure to the stabilized residual-based FE methods, such as SUPG. As an almost free consequence of this reformulation we get the conservation property of the EDG (and HDG) schemes on dual volumes and some insights into the structure of the flux boundary conditions for the EDG scheme. In section 5, within the context of the EDG approach, the important concept of the local linearized problem for the correction is introduced. Thereafter we show that the lowest order approximation to this local problem leads to the SUPG discretization with one of the standard definitions of the SUPG stabilization matrix. Section 6 contains discussion of the obtained results in a wider