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Analysis of stretched grids as buffer zones in simulations of wave propagation

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ABSTRACT

A zone of increasingly stretched grid is a robust and easy-to-use way to avoid unwanted reflections at artificial boundaries in wave propagating simulations. In such a buffer zone there are two main damping mechanisms, dissipation and under-resolution that turns a traveling wave into an evanescent wave. We present analysis in one and two space dimensions showing that evanescent decay through under-resolution is a very efficient way to damp waves. The analysis is supported by numerical computations.

1. Introduction

Non-reflecting boundary conditions are of vital importance in numerical simulations of convective flows and the research on non-reflecting boundary conditions has long going traditions. In general exact non-reflecting boundary conditions are global, both in space and time. There are also many local non-reflecting boundary conditions. They can be divided into three main types namely, local approximations of exact non-reflecting boundary conditions, see e.g. [7,8,11], buffer zones, see e.g. [12,1] and Perfectly Matched Layers, see e.g. [3,2].

In [4] an overview of non-reflecting boundary conditions for compressible flows is given. In the paper the vast development of non-reflecting boundary conditions for linear problems is pointed out, whereas there is a need for enhanced development and knowledge of the performance of non-reflecting boundary conditions for non-linear flows such as turbulent shear flows.

Most buffer zones involve a stretching of the grid, but damping can also be enhanced by a forcing function or artificial viscosity. The popularity of buffer zones lays in their simplicity. They are easy to implement and in most cases, the change in the time step restriction due to stability is small. However, unwanted reflections from buffer zones can occur either at the entrance at the buffer zones, due to e.g. grid stretching or sudden increase of artificial viscosity and forcing functions, or reflections at the outflow boundary due to too low damping within the buffer zone.

In [1] reflections from buffer zones due to under-resolution of outgoing waves, and the influence of different orders of artificial viscosity terms on the damping of reflections are studied. It is shown by numerical experiments for linear hyperbolic systems that artificial viscosity terms based on high-order undivided difference substantially reduce the reflections from the buffer zone. For a dispersive scheme, under-resolution of a wave can be seen as equivalent to lowering the phase

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speed of the wave. Karni [14] suggested a slowing-down operator and a similar concept was investigated as the acoustic black hole layer boundary condition in [16].

A more simplistic approach is to use grid stretching in the buffer zone together with the same numerical scheme as in the interior computational domain. The main difficulty with this type of boundary zone is to determine how large grid stretching that should be applied and the number of grid points in the boundary zone. Often the choice is based on a rule of thumb from previous knowledge of the particular code used. Hence, a better understanding of the propagation and dissipation of waves on a stretched grid decreases the time spent on determining suitable buffer zones in a practical simulation.

In a series of papers Vichnevetsky et al., [19–21], analyzed the influence of grid stretching on wave propagation in one space dimension. The analysis is based on the discrete advection equation, where waves and wave packets are imposed as initial data and the discrete equation is Fourier transformed in space or in time. The conclusions from the papers can be summarized as follows, see [21]: a slowly and uniformly changing mesh creates no scattering as long as waves are well resolved, but a sharp change in a otherwise uniformly changing mesh creates scattering also for a well resolved wave. Hence, when a wave-like structure enters a region with a smoothly stretched grid, the reflection will be very small. Our experience agrees well with this conclusion. However, for wave packets the reflections due to a group velocity of opposite sign are of the same order as the amplitude of the wave packet, see [9,15].

In this paper, we focus on buffer zones based on pure grid stretching with a constant numerical viscosity coefficient. The focus of the paper is to analyze the damping mechanisms for waves in stretched grids. A main result is that damping from grid stretching is much more efficient than damping by dissipation when the grid stretching turns the propagating waves into evanescent waves. A similar result was briefly discussed in [13], but is studied in more detail in this paper. With this understanding, reflections of upstream waves originating from the outflow boundary can be avoided. The results of the analysis can in turn be used to formulate estimates on the number of cells needed in a buffer zone, for a given system and grid stretching ratio.

Two approaches can be taken when analyzing the effect of varying grid spacing on wave propagation. Either, the frequency in time is kept constant via imposing temporally periodic boundary data. This approach leads to analysis of a discrete boundary value problem. The other alternative, which so far has been the most common approach, is to impose a wave as periodic initial data on an infinite domain, see [6,10,20]. As will be seen, the two approaches are naturally equal for the continuous problem when the model includes only advection. For the semi-discrete equation, however, the two approaches lead to different insights.

To begin with we analyze semi-discrete scalar advection and advection–diffusion equations, respectively, where time periodicity is imposed as boundary data in a one dimensional setting. We find that physically propagating waves turn into evanescent waves when the grid resolution is low. We analyze the decay rate, and a main contribution of this work is the conclusion that the amplitude of waves is very efficiently reduced in the evanescent regime. This type of decay is compared with decay due to viscous damping, and we find that evanescent decay is a much more efficient way to reduce the amplitude of physical waves. In many cases there are also high frequency spurious waves present, and viscous damping can be important for such waves. We also extend the analysis to hyperbolic systems in two space dimensions, and apply the results to the linearized Euler equations in an aero-acoustic setting. Important contributions in the paper are the precise expressions for decay of waves, which can be used for determining the thickness of buffer zones both in one and two space dimension. Numerical computations demonstrate the validity of the analysis.

2. Background

In the main part of this paper, we analyze the discrete solution of linear hyperbolic equations in terms of boundary value problems. In [20,21] as well as in [6] the discrete solutions are studied in form of an initial value problem. In order to compare the insights from the two approaches, we describe the initial value approach in this section for completeness.

We consider

$$u_t + cu_x = 0, \quad -\infty < x < \infty, \tag{1}$$

with periodic initial data, that is

$$u(x,0) = e^{-i\xi x}. \tag{2}$$

The solution of the continuous problem is

$$u(x,t) = e^{i(\omega t - \xi x)}, \tag{3}$$

where the temporal frequency ω and the spatial wave number ξ are related via the dispersion relation

$$\omega = c\xi. \tag{4}$$

We discretize in space using second order accurate central differences, yielding

$$(v_j)_t + cD_0 v_j = 0. \tag{5}$$