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Multigrid methods for cubic spline solution of two point (and 2D) boundary value problems

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ABSTRACT

In this paper we propose a scheme based on cubic splines for the solution of the second order two point boundary value problems. The solution of the algebraic system is computed by using optimized multigrid methods. In particular the transformation of the stiffness matrix essentially in a block Toeplitz matrix and its spectral analysis allow to choose smoothers able to reduce error components related to the various frequencies and to obtain an optimal method. The main advantages of our strategy can be listed as follows: (i) a fourth order of accuracy combined with a quadratic conditioning matrix, (ii) a resulting matrix structure whose eigenvalues can be compactly described by a symbol (this information is the key for designing an optimal multigrid method). Finally, some numerics that confirm the predicted behavior of the method are presented and a discussion on the two dimensional case is given, together with few 2D numerical experiments.

1. Introduction

The study and the analysis of the second order two point boundary value problems are important in themselves and because they can provide crucial information with regard to more general partial differential problems [3,12]. In fact, depending on the problem which is to be solved, a high regularity to the numerical solution may be required, for example the continuity of the mathematical fluxes, the latter being an important property in computational fluid dynamic problems. Among the numerical methods intrinsically conservative (i.e. able to guarantee the $\mathcal{C}^1$ property of the numerical solution) we can quote the finite volumes, the Galerkin discontinuous finite elements and the methods based on Csplines [9,5,11,15,16]. It is well-known that the algebraic systems obtained by finite elements (FE) approximation of the elliptic differential operators are ill conditioned and, moreover for real problems, very large. Due to this, it appears suitable to adopt iterative techniques for their solution, for example to use methods based on Krylov spaces or multigrid methods that are able to calculate the algebraic solutions with a convergence ratio independent of the system dimension [10,7]. In this paper we will present a scheme based on Csplines (thus globally of $\mathcal{C}^1$ class), where the solution of the system is obtained by designing optimized multigrid schemes, following the ideas in [17,20]. The optimization will be obtained by the analysis of the functions to which asymptotically the eigenvalues of the stiffness matrices, suitably transformed, converge. Indeed,

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up to a low rank correction, the resulting structure is a banded Toeplitz structure whose generating function or symbol is Hermitian 2×2 and nonnegative definite so that the resulting spectrum is asymptotically described by the two eigenvalues of the symbol [18,23,21]. In the final section the computational efficiency of our approach will be tested and discussed.

The technique is adaptable naturally to elliptic two dimensional second order boundary value problems. In such a case we encounter two level structures, which up to low rank corrections, have a (block) two level Toeplitz pattern and are banded in a two level sense. As in the one dimensional setting, both the spectral analysis and the design of the multigrid solver can be achieved by a careful study of analytical properties of the associated symbol, which will be a bivariate, nonnegative definite matrix valued function, whose entries are bivariate trigonometric polynomials.

The paper is organized as follows. In Section 2 we describe the one dimensional differential problem and we derive in detail the structure of the associated coefficient matrix by using a scheme based on Csplines. Section 3 is devoted to the spectral analysis of our matrix structures, in terms of conditioning, extreme eigenvalues, and global distribution results. The spectral analysis is then used as a guideline for designing optimized multigrid techniques in Section 4. Section 5 is devoted to numerical experiments while in Section 6 we sketch the two dimensional case, by showing that our numerical approach can be extended fruitfully to higher dimensions. In Section 7 we draw conclusions and we report a few open problems.

2. Test problem

The reference problem of our study is:

$$-u''(x) = f(x), \quad x \in \Omega = [0,1], \tag{1}$$

with boundary conditions $u(x=0) = u(x=1) = 0$ and $f(x)$ known function. In order to clarify the structure of the global stiffness matrix A^g of the final system

$$A^g u = F, \tag{2}$$

the criteria followed in its construction are now quickly recalled.

Given the standard Galerkin weak formulation of the problem (1)

$$\int_\Omega u'(x)v'(x)dx = \int_\Omega f(x)v(x)dx, \tag{3}$$

we approximate it by an FE technique choosing a proper finite dimensional Sobolev space $V_h \subset H_0^1(\Omega)$ to which both the numerical solution u_h and the test functions have to belong.

The process of numerical integration is carried out in the unit element $[0,1]$, $0 \le \xi \le 1$, after that the domain Ω has been partitioned into elements $[x_i, x_{i+1}]$, $x_{i+1} = x_i + h$, with constant diameter h and that the transformation $\xi = \frac{x - x_i}{h}$ has been applied to each element.

The base functions of the V_h space are Cspline polynomials. It is well-known that, assigned in $N+1$ points x_i the values of $g(x_i)$ and $g'(x_i)$ of a function $g(x)$, the Hermite interpolating polynomial $p(x)$ of $g(x)$ can be written as the sum of two different polynomials $q_j(x)$ and $r_j(x)$:

$$p(x) = \sum_{j=0}^{N} q_j(x)g(x_j) + \sum_{j=0}^{N} r_j(x)g'(x_j), \tag{4}$$

where $q_j(x)$ and $r_j(x)$ are defined by means of the Lagrange polynomials $l_j(x)$:

$$q_j(x) = \left[1 - 2l_j'(x_j)(x - x_j)\right] l_j^2(x), \tag{5}$$

$$r_j(x) = (x - x_j) l_j^2(x), \tag{6}$$

cf. [22]. For every j, k the following relationships are valid:

$$U_j(x_k) = \delta_{j,k}, \tag{7}$$

$$V_j'(x_k) = \delta_{j,k}, \tag{8}$$

$$V_j(x_k) = 0, \tag{9}$$

$$U_j'(x_k) = 0, \tag{10}$$

and hence, by considering (6), (5), and (4), we can deduce that the solution $u_h \in \mathcal{C}^1$ can be written as a combination of the nodal values U_i and U_i', and of the basic functions $\varphi_i(x)$ and $\psi_i(x)$, i.e.,

$$u_h(x) = \sum_{K_i} \left(\varphi_i(x)U_i + \psi_i(x)U_i'\right), \tag{11}$$

where $\varphi_i(x)$ and $\psi_i(x)$ are related to $q_j(x)$ and $r_j(x)$, respectively.