Applied Numerical Mathematics 104 (2016) 141–157

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

A hybrid recursive multilevel incomplete factorization preconditioner for solving general linear systems

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ARTICLE INFO

Article history:
Available online 15 February 2016

Keywords:
Linear systems
Iterative solvers
Preconditioners
Sparse approximate inverse methods
Multilevel reordering algorithms

ABSTRACT

In this paper we introduce an algebraic recursive multilevel incomplete factorization preconditioner, based on a distributed Schur complement formulation, for solving general linear systems. The novelty of the proposed method is to combine factorization techniques of both implicit and explicit types, recursive combinatorial algorithms, multilevel mechanisms and overlapping strategies to maximize sparsity in the inverse factors and consequently reduce the factorization costs. Numerical experiments demonstrate the good potential of the proposed solver to precondition effectively general linear systems, also against other state-of-the-art iterative solvers of both implicit and explicit form.

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1. Introduction

Krylov subspace methods may be considered the method of choice for solving large and sparse systems of linear equations arising from the discretization of (systems of) partial differential equations on modern parallel computers. This class of algorithms are iterative in nature. At every step k, they compute the approximate solution x_k of a linear system $Ax = b$ from the Krylov subspace of dimension k

$$K_k(A, b) = span\{b, Ab, A^2b, \dots, A^{k-1}b\},$$

according to different criteria for each given method. The computation requires matrix–vector products with the coefficient matrix A plus vector operations, thus potentially reducing the cumbersome costs of sparse direct solvers on large problems, especially in terms of memory. All of the iterative Krylov methods converge rapidly if A is somehow close to the identity. Therefore, it is common replacing the original system $Ax = b$ by

$$M^{-1}Ax = M^{-1}b, \tag{1}$$

or

$$AM^{-1}y = b, \quad x = M^{-1}y, \tag{2}$$

for a nonsingular matrix $M \approx A$. Systems (1) and (2) are referred to as *left* and *right preconditioned* systems, respectively, and M as the *preconditioner matrix*. In the case M is factorized as the product of two sparse matrices, $M = M_1M_2$, like in the Hermitian and positive definite case, one might solve the modified linear system

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http://dx.doi.org/10.1016/j.apnum.2015.12.007
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$$M_1^{-1}AM_2^{-T}y = M_1^{-1}b, \;\; x = M_2^{-T}y. \tag{3}$$

If one may choose M so that $M^{-1}A$, AM^{-1} or $M_1^{-1}AM_2^{-T}$ approximate the identity, and linear systems with M or with M_1 and M_2 as coefficient matrices are easy to invert, it is more efficient to apply a Krylov subspace method to the modified linear system.

Optimal analytic preconditioners based on low order discretizations, nearby equations that are simple to solve, or similar ideas have been proposed in the literature for specific problems. However, the problem-specific approach is generally sensitive to the characteristics of the underlying operator and to the details of the geometry. In this study, we pursue an algebraic approach where the preconditioner M is computed only from the coefficient matrix A. Although not optimal for any specific problem, algebraic methods are universally applicable, they can be adapted to different operators and to changes in the geometry by tuning a few parameters, and are well suited for solving irregular problems defined on unstructured grids.

Roughly speaking, most of the existing techniques can be divided into either implicit or explicit form. A preconditioner of *implicit* form is defined by any nonsingular matrix $M \approx A$, and requires to solve an extra linear system with M at each step of an iterative method. The most important example in this class is represented by the Incomplete LU decomposition (ILU), where M is implicitly defined as $M = \bar{L}\bar{U}$, $\bar{L}$ and $\bar{U}$ being triangular matrices that approximate the exact L and U factors of A according to a prescribed dropping strategy adopted during the Gaussian elimination process. These methods are considered amongst the most reliable in a general setting. Well known theoretical results on the existence and the stability of the factorization can be proved for the class of M-matrices [35], and recent studies are involving more general matrices, both structured and unstructured. The quality of the factorization on difficult problems can be enhanced by using several techniques such as reordering, scaling, diagonal shifting, pivoting and condition estimators (see e.g. [16,44,36,7,9]). As a result of this active development, in the last years successful results are reported with ILU-type preconditioners in many areas that were of exclusive domain of direct solution methods like in circuits simulation, power system networks, chemical engineering plants modelling, graphs and other problems not governed by PDEs, or in areas where direct methods have been traditionally preferred, like in structural analysis, semiconductor device modelling and computational fluid dynamics applications (see e.g. [41,6,1,34,43]). One problem with ILU-techniques is the severe degradation of performance observed on vector, parallel and GPUs machines, mainly due to the sparse triangular solves [33]. In some cases, reordering techniques may help to introduce nontrivial parallelism. However, parallel orderings may sometimes degrade the convergence rate, while more fill-in diminishes the overall parallelism of the solver [17].

Explicit preconditioning tries to mitigate such difficulties by approximating directly A^{-1}, as the product M of sparse matrices, so that the preconditioning operation reduces to forming one (or more) sparse matrix–vector product, and consequently the application of the preconditioner may be easier to parallelize and numerically stable. Some methods can also perform the construction phase in parallel [23,10,26,37,38]; additionally, on certain indefinite problems with large nonsymmetric parts, the explicit approach can provide better results than ILU techniques (see e.g. [14,8,24]). In practice, however, some questions need to be addressed. The computed matrix M could be singular, and the construction cost is typically much higher than for ILU-type methods, especially for sequential runs. The main issue is the selection of the non-zero pattern of M. The idea is to keep M reasonably sparse while trying to capture the 'large' entries of the inverse, which are expected to contribute the most to the quality of the preconditioner. On general problems it is difficult to determine the best structure for M in advance, and the computational and storage costs required to achieve the same rate of convergence of preconditioners given in implicit form may be prohibitive in practice.

In this study, we present an algebraic multilevel solver for preconditioning general nonsymmetric linear systems which attempts to combine characteristics of both approaches. Assuming that the matrix A admits the factorization $A = LU$, with L a unit lower and U an upper triangular matrix, our method approximates the inverse factors L^{-1} and U^{-1}. Sparsity in the approximate inverse factors is maximized by employing recursive combinatorial algorithms. Robustness is enhanced by combining the factorization with recently developed overlapping strategies and by using efficient local solvers.

The paper is organized as follows. In Section 2 we describe the proposed multilevel preconditioner. In Section 3 we show how to combine our preconditioner with overlapping strategies, and in Section 4 we assess its overall performance by showing several numerical experiments on realistic matrix problems, also against other state-of-the-art solvers. Finally, in Section 5 we conclude the study with some remarks and perspectives for future work.

2. The AMES solver

Let

$$Ax = b \tag{4}$$

be a $n \times n$ general linear system with nonsingular, possibly indefinite and nonsymmetric matrix $A = \{a_{ij}\} \in \mathbb{R}^{n\times n}$, and vectors $x, b \in \mathbb{R}^n$. We assume that A admits for a triangular decomposition

$$A = LU$$

and we precondition system (4) as

$$M_L A M_U y = M_L b$$

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