Applied Numerical Mathematics 104 (2016) 182–203

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Numerical solution of optimization problems for semi-linear elliptic equations with discontinuous coefficients and solutions

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ARTICLE INFO

Article history:
Available online 11 February 2016

Keywords:
Semi-linear elliptic equations
Optimal control problem
Numerical method
Iterative method
Operator

ABSTRACT

In this work we consider optimization problems for processes described by semi-linear partial differential equations of elliptic type with discontinuous coefficients and solutions (with imperfect contact matching conditions), with controls involved in the coefficients. Finite difference approximations of optimization problems are constructed. For the numerical implementation of finite optimization problems differentiability and Lipshitz-continuity of the grid functional of the approximating grid problems are proved. An iterative method for solving boundary value problems of contact for PDEs of elliptic type with discontinuous coefficients and solutions is developed and validated. The convergence of the iterative process is investigated. And the convergence rate of iterations (with calculated constants) is estimated.

0. Introduction

Optimization mathematical models for systems with distributed parameters described by the equations of mathematical physics (PDEs) is one of the most complicated class of problems in the optimization theory, especially for nonlinear control problems. A special interest both for practical and theoretical points of view is related to a physical and mathematical formulation of optimal control problems, in which, due to a nature of a studied physical process, the states are described by nonlinear PDEs with discontinuous coefficients, and moreover, originally in their physical and mathematical formulation, solutions of PDEs admit discontinuities [9,11,8,10]. Such problems arise in the mathematical modeling and optimization of heat transfer, diffusion, filtration, elasticity, etc., in a study of inverse problems and of optimal control problems for equations of mathematical physics in multilayered media.

Before solving optimal control problems numerically, they have to be approximated by problems of a simpler nature, specifically, by "finite-dimensional optimization problems" (see [14]). One of the most convenient, universal, and widespread techniques for finite-dimensional approximation as applied to optimal control problems is the grid method [9,11,8,10]. An overview of works addressing the foundations of the general theory and methods of stability, the approximations of optimal control problems, and results obtained in this area can be found, for example, in [14,2]. Note that finite difference schemes for equations with discontinuous coefficients, but with continuous fluxes and solutions (with perfect-contact matching condition) were constructed and examined in [9,8] for PDEs having classical solutions of some degree of smoothness. The

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[1] This work was supported by a grant of the President of Russian Federation for state support of young Russian scientists – Ph.D. (MK-4147.2015.1).

http://dx.doi.org/10.1016/j.apnum.2015.12.006

convergence of difference schemes for parabolic equations with discontinuous coefficients and solutions in classical formulations with sufficiently smooth solutions was analyzed in [12,13].

In the present work, in the field related to [14,4–7], we consider nonlinear optimal control problems governed by semi-linear elliptic equations in inhomogeneous anisotropic media with discontinuous coefficients and solutions (states) and with matching boundary conditions of the imperfect-contact type [9,3]. The coefficients in the right-hand side of equation are used as a control function. We construct and investigate finite difference approximations of optimization problems. Note that issues of well-posedness of the optimization problems and their approximations, convergence of the approximations with respect to the state and the cost functional; weak convergence of the approximations with respect to the control; and the regularization of the approximations using Tikhonov regularization are examined by [4,7], see Section 3 and Section 4. Observe also that development of efficient numerical methods for solving finite-dimensional grid optimal control problems is not investigated in these works.

There are two steps to solving finite-dimensional grid optimal control problems. First, we have the problem of constructing effective, high-precision approximate methods of solving boundary value problems for PDEs with discontinuous coefficients and solutions – problems for the state. In particular, we have the problem of constructing effective convergent iterative methods for the solution of this class of problems for PDEs, as well as the problems of development and implementation of finite-dimensional approximations (see, for example, [9], [14,4–7]) of iterative problems at each iteration step. Development of methods for solving contact problems for PDEs with discontinuous coefficients and solutions is an independent and important issue.

In the present work we develop and validate an iterative method for solving grid boundary value problems of contact for elliptic equations with discontinuous coefficients and solutions. A convergence of the iterative process is investigated. The strong convergence of the iterative process to a unique solution to the difference boundary value problem is proved and the convergence rate of iterations (with calculated constants) is estimated. As a result, the numerical solution of these problems can be effectively implemented on the basis of the developed iterative method (with iterations on the inner boundary where the coefficients and solutions are discontinuous) in combination, for example, with the difference method for solving some already traditional "independent" boundary value problems arising in each contacting subdomain inside the composite integration domain. Note that since the developed iterative process converges strongly, and, moreover, the convergence properties of the approximating grid optimal control problems to the original optimization problem are proved in our previous articles [4,7] (see also Remark 1 of the current paper), it follows that the discrete approximate states (constructed by the iterative procedure) converge to the state associated to the original optimal control.

For the second step it is necessary to develop numerical algorithms for minimizing a cost functional, depending on a state of the system and a control. To this aim differentiability and Lipshitz-continuity of the grid functional of the approximating grid problems are proved in the present work. Effective procedures for calculating gradients of minimized functionals using the solutions of direct problems for the state and adjoint problems are obtained.

1. Formulation of optimal control problems

Let $\Omega = \{r = (r_1, r_2) \in \mathbf{R}^2 : 0 \le r_\alpha \le l_\alpha, \alpha = 1, 2\}$ be a rectangle in $\mathbf{R}^2$ with a boundary $\partial\Omega = \Gamma$. The domain Ω is divided by the line $r_1 = \xi$, where $0 < \xi < l_1$ (by the internal interface $\overline{S} = \{r_1 = \xi,\ 0 \le r_2 \le l_2\}$, where $0 < \xi < l_1$) into the left $\Omega_1 \equiv \Omega^- = \{0 < r_1 < \xi,\ 0 < r_2 < l_2\}$ and right $\Omega_2 \equiv \Omega^+ = \{\xi < r_1 < l_1,\ 0 < r_2 < l_2\}$ subdomains with boundaries $\partial\Omega_1 \equiv \partial\Omega^-$ and $\partial\Omega_2 \equiv \partial\Omega^+$. Thus, Ω is the union of Ω_1 and Ω_2 and the interior points of the interface $\overline{S}$ between Ω_1 and Ω_2, while $\partial\Omega$ is the outer boundary of Ω. Let $\overline{\Gamma}_k$ denote the boundaries of Ω_k without S, $k = 1, 2$. Therefore $\partial\Omega_k = \overline{\Gamma}_k \cup S$, where Γ_k, $k = 1, 2$ are open nonempty subsets of $\partial\Omega_k$, $k = 1, 2$; and $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega = \Gamma$. Let n_α, $\alpha = 1, 2$ denote the outward normal to the boundary $\partial\Omega_\alpha$ of Ω_α, $\alpha = 1, 2$. Let $n = n(x)$ be a unit normal to S at a point $x \in S$, directed, for example, so that n is the outward normal on S with respect to Ω_1; i.e., n is directed inside Ω_2. While formulating boundary value problems for states of control processes below, we assume that S is a straight line across which the coefficients and solutions of the problems are discontinuous, while being smooth within Ω_1 and Ω_2.

Assume that the conditions imposed on a controlled physical process are such that it can be modeled in the domain $\Omega = \Omega_1 \cup \Omega_2 \cup S$, which consists of two subdomains Ω_1 and Ω_2, and the separating internal boundary S, by the following Dirichlet problem for a semi-linear elliptic equation with discontinuous coefficients and solution: Find a function $u(x)$, defined on $\overline{\Omega}$ that satisfies in Ω_1 and Ω_2 the equations:

$$Lu(x) = -\sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right) + d(x)q(u) = f(x), \quad x \in \Omega_1 \cup \Omega_2, \tag{1}$$

and the conditions

$$u(x) = 0, \quad x \in \partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2, \quad \left[k_1(x)\frac{\partial u}{\partial x_1}\right] = 0, \quad G(x) = \left(k_1(x)\frac{\partial u}{\partial x_1}\right) = \theta(x_2)[u], \quad x \in S,$$

where $u(x) = \begin{cases} u_1(x), & x \in \Omega_1; \\ u_2(x), & x \in \Omega_2, \end{cases}$ $\quad q(\xi) = \begin{cases} q_1(\xi_1), & \xi_1 \in \mathbf{R}; \\ q_2(\xi_2), & \xi_2 \in \mathbf{R}, \end{cases}$