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Initial value problems with retarded argument solved by iterated quadratic splines

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A B S T R A C T

In this paper we propose a new iterative numerical method for initial value problems of first and second order involving retarded argument. The method uses a quadratic spline interpolation procedure activated at each iterative step. The convergence of this method of iterated splines is theoretically proven and tested on some numerical examples.

1. Introduction

In this paper we construct a new iterative numerical method in order to approximate the solution of initial value problems for first and second order differential equations with retarded argument. The existing numerical methods proposed for initial value problems associated to first and second order ordinary differential equations and differential equations with retarded argument are based on: Runge–Kutta procedures (see [1,2,16,29,34,38]), Nyström techniques (see [18]), power series (see [33] and [39]), iterative analytic-numeric methods like variational iteration, Adomian decomposition, and homotopy perturbation (see [9,17,31,40]), collocation methods (see [5,7,13,14,32]), Adams procedures and divided differences (see [19] and [23]), Birkhoff interpolation (see [11]), rational approximation (see [21]), spline functions methods (see [7,27,30,15]), B-spline scaling functions (see [24]), quadrature collocation based on local radial basis functions (see [12]), pseudospectral tau and Lanczos methods (see [36]). The method of successive interpolations based on Birkhoff cubic splines with deficiency 3 is presented in [4] for second order functional differential equations. In [3] the method of successive interpolations uses natural cubic splines for the numerical solution of first order functional differential equations. In [26] and [25] the use of quadratic splines was involved in the spline functions method applied to first order ODEs.

The proposed numerical method developed here is based on quadratic splines and will be applied to the following kind of initial value problems:

$$\begin{cases} x'(t) = f(t, x(t), x(\varphi(t))), \quad t \in [a,b] \\ x(a) = x_0 \end{cases} \tag{1}$$

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and

$$\begin{cases} x''(t) = f(t, x(t), x(\varphi(t))), & t \in [a,b] \\ x(a) = x_0, \ x'(a) = x_0' \end{cases} \tag{2}$$

where $f \in C([a,b] \times \mathbb{R} \times \mathbb{R})$ and $\varphi \in C[a,b]$ with $a \le \varphi(t) \le t$, for all $t \in [a,b]$. This method combines the Picard's iterative technique with a quadratic spline interpolation procedure applied in each iterative step. As a particular case we mention the well-known nonlinear pantograph equation with vanishing retarded argument $\varphi(t) = \beta t$, $\beta \in (0,1)$:

$$\begin{cases} x'(t) = f(t, x(t), x(\beta t)), & t \in [a,b] \\ x(a) = x_0. \end{cases} \tag{3}$$

The classical linear pantograph equation $x'(t) = A \cdot x(t) + B \cdot x(\beta t)$ appears in [28], modelling current collection by electric locomotive's pantograph. The equation is studied in [31,36], and [40], the numerical methods being developed for the approximate solution using variational iteration, pseudospectral Lanczos methods with Chebyshev polynomials, and homotopy perturbation, respectively.

The well-known performances of the existing numerical methods (such as continuous Runge–Kutta, collocation, spline functions methods, and polynomial expansions like: Adomian decomposition, variational iteration, homotopy perturbation, power series, series of orthogonal polynomials) for initial value problems associated to functional differential equations are widely disseminated in a rich literature (see [1,2,5,6,25,27,34], and references therein). In addition to this, we develop in this paper the new method of iterated splines based on Picard's technique of successive approximations and applying in each iterative step suitable quadrature rules and spline interpolation procedures. Our intention is to show that such iterative numerical method could be effective when the quadrature rule and the spline are adequately chosen. The paper is organized as follows: in Section 2 we present the convergence properties of the quadratic spline interpolation procedure and the construction of the proposed iterative algorithm. The convergence of the proposed numerical method is proven in Section 3 and it is tested on some numerical experiments in Section 4. In Section 3 we introduce a new type of numerical stability that is appropriate for iterative methods, namely the numerical stability with respect to the choice of the first iteration. For this it is proven that the proposed numerical method is numerically stable. The presented numerical experiments confirm the numerical stability. Some concluding remarks are pointed out in Section 5.

2. The quadratic splines and the iterative algorithm

In [37] it is proposed a quadratic spline for interpolating given values $y_0, y_1, \ldots, y_n$ on a set of corresponding knots $x_0, x_1, \ldots, x_n \in [a,b]$, with $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. This quadratic spline $s : [a,b] \to \mathbb{R}$, $s \in C^1[a,b]$ is generated by the interpolation conditions $s(x_i) = y_i$, $i = \overline{0,n}$, $s'(x_0) = m_0$, and has the restrictions s_i, to the subintervals $[x_{i-1}, x_i]$, $i = \overline{1,n}$:

$$s_i(x) = \frac{m_i - m_{i-1}}{2h_i} \cdot (x - x_{i-1})^2 + m_{i-1} \cdot (x - x_{i-1}) + y_{i-1}, \ \forall x \in [x_{i-1}, x_i], \ \forall i = \overline{1,n} \tag{4}$$

where $h_i = x_i - x_{i-1}$, $i = \overline{1,n}$. The smoothness conditions $s \in C[a,b]$, $s \in C^1[a,b]$ lead to $s_i(x_i) = y_i$, $\forall i = \overline{1,n}$. It is easy to see that $s_i'(x_{i-1}) = m_{i-1}$, $s_i'(x_i) = m_i$, $\forall i = \overline{1,n}$, and by the conditions $s_i(x_i) = y_i$, $i = \overline{1,n}$, we get the relations:

$$y_i - y_{i-1} = (m_i - m_{i-1}) \cdot \frac{h_i}{2} + m_{i-1} h_i$$

which can be written in the recurrent form:

$$m_i = \frac{2}{h_i} \cdot (y_i - y_{i-1}) - m_{i-1}, \quad \forall i = \overline{1,n}. \tag{5}$$

Therefore we infer that the values $y_0, y_1, \ldots, y_n$ and m_0 uniquely determine the C^1 smooth quadratic spline interpolating the values y_i, $i = \overline{0,n}$. In the case that $y_i = f(x_i)$, $i = \overline{0,n}$, for $f : [a,b] \to \mathbb{R}$, we can say that s interpolates f on the knots x_i, $i = \overline{0,n}$. With respect to error estimation of this interpolation, in [37] error estimates for the cases $f \in C^3[a,b]$ and $f \in C^4[a,b]$ in the context of uniform grids are obtained. Here we derive the error estimates in the cases of lower smooth functions for arbitrary grids.

Theorem 1. *(i) If $f : [a,b] \to \mathbb{R}$ is a continuous function such that $f(x_i) = y_i$, $\forall i = \overline{0,n}$ and if $s : [a,b] \longrightarrow \mathbb{R}$, $s \in C^1[a,b]$ is the quadratic spline interpolating f, then for any choice of the value m_0 the error estimate is:*

$$|s(x) - f(x)| \le \left(1 + \frac{h}{\underline{h}}\right) \cdot \omega(f,h) + O\left(h^2\right), \quad \forall x \in [a,b] \tag{6}$$

where $h = \max\{h_i : i = \overline{1,n}\}$, $\underline{h} = \min\{h_i : i = \overline{1,n}\}$, and

$$\omega(f,h) = \sup\{\left|f(x) - f\left(x'\right)\right| : x, x' \in [a,b], \ \left|x - x'\right| \le h\}$$