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A path-conservative Osher-type scheme for axially symmetric compressible flows in flexible visco-elastic tubes

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A B S T R A C T

Flexible tubes are widely used in modern industrial hydraulic systems as connections between different components like valves, pumps and actuators. For the design and the analysis of the temporal behavior of a hydraulic system, one therefore needs an accurate mathematical model that describes the fluid flow in a compliant duct. Hence, in this paper we want to model the fluid–structure-interaction (FSI) problem given by the axially symmetric flow of a compressible barotropic fluid that flows through flexible tubes made of vulcanized rubber. The material of the tube can be described by using a visco-elastic rheology, which takes into account the strain relaxation of the material. The resulting mathematical model consists in a one-dimensional system of nonlinear hyperbolic partial differential equations (PDE) with non-conservative products and algebraic source terms. To solve this system numerically, we apply the DOT method, which is a generalized path-conservative Osher-type Riemann solver for conservative and non-conservative hyperbolic PDE recently proposed in [23] and [22].

We provide numerical evidence that the proposed DOT Riemann solver is *well-balanced* for the governing PDE system under consideration. The method is compared to available quasi-exact solutions of the Riemann problem in the case of an elastic wall described by the Laplace law. It is also compared to available experimental data and exact solutions obtained in the frequency domain for a linear visco-elastic wall behavior. In all cases under investigation the proposed path-conservative finite volume scheme based on the DOT Riemann solver is able to produce very accurate results.

1. Introduction

Nowadays there exist many important industrial hydraulic systems that require an accurate mathematical modeling of the system behavior already at the design stage. Especially in the automotive industry there are, for example, the braking systems and the fuel injection into the combustion engine. As these systems get more and more complex, and since they should be more and more efficient due to increased safety standards or an increasingly restrictive environmental legislation, one needs to understand each single component as well as the entire hydraulic system and the interaction between the different components. When one wants to understand the dynamic behavior of the system already at the design stage, an important tool is its mathematical modeling and the subsequent numerical simulation.

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Since flexible tubes are used in many hydraulic systems for the connection of different components, it is very important to understand and simulate their behavior. Therefore one needs a model for the physical effects occurring in the fluid as well as in the surrounding elastic tube. The fluid can be modeled by the one-dimensional cross-sectionally averaged compressible Euler equations in a duct with variable cross section, see [31,44]. Hooke's law has been used in the past [24] for the description of the elasticity of the tube wall, as well as the law of Laplace [13]. But when comparing the numerical results obtained with these simplified laws to experimental results of a flexible polymer tube, one can see that there is not enough dissipation in the system. So the next step was to use a visco-elastic model. Such models consist of dashpots and springs, so that they can reproduce the elastic as well as the viscous behavior of the wall material. Such models are for example described in [32].

In this paper we therefore want to model the fluid–structure-interaction (FSI) problem given by the axially symmetric flow of a compressible barotropic fluid through flexible polymer tubes. Hence, we present two different elasticity models—a simple Laplace law and a the widely used Maxwell model that is capable of reproducing visco-elastic effects of the polymer material [27]. In combination with the Euler equations for the flow of a compressible fluid in flexible tubes, we get a system of non-conservative hyperbolic partial differential equations (PDE).

To solve this non-conservative PDE system we employ the family of path-conservative schemes developed by Castro & Parés and co-workers [10,43,7,25,8,37,12,18,19,17], which are based on the theory of Dal Maso, Le Floch and Murat [36] on weak solutions for hyperbolic PDE with non-conservative products. The family of path-conservative schemes can be seen as a generalization of the weak formulation of the Roe method proposed by Toumi [46] and of the wave-propagation algorithm of LeVeque [35,34] for PDE with source terms. We would like to stress that there are still unresolved problems related to the choice of the path for general non-conservative PDE, which have been discussed in detail in [11].

In the following, we use a generalized version of the Riemann solver of Osher and Solomon [42] for conservative and non-conservative hyperbolic PDE recently proposed by Dumbser and Toro [23,22] (DOT). Note that the DOT solver can be further simplified in the context of polynomial viscosity methods (PVM), as recently described in [9]. We also would like to refer the reader to the successful use of path-conservative finite volume schemes based on the DOT Riemann solver in the context of computational hemodynamics, see the work of Müller & Toro et al. [39–41,38], where very recently the flow of an incompressible fluid in a network of compliant blood vessels has been considered. The present scheme is meant to be an alternative discretization of compressible flows in compliant ducts based on *explicit* path-conservative Godunov-type finite volume schemes, compared to the *semi-implicit* finite-volume and finite-difference discretization of such flows presented in [20]. The theoretical framework of these nonlinear semi-implicit schemes was presented in the papers by Brugnano and Casulli [4–6] and Casulli and Zanolli [14,15]. From the conceptual point of view, there are several important differences between the semi-implicit approach [20], and the path-conservative Godunov-type method presented here, which we briefly discuss in the following. The semi-implicit finite volume scheme [20] has to obey only a very mild CFL condition that is based on the *flow velocity*, while the explicit path-conservative scheme used here has to obey a more stringent CFL stability condition based on the *effective sound speed* of the coupled fluid–structure system. This makes the semi-implicit scheme [20] particularly suitable and computationally efficient for low Mach number and quasi-incompressible flows. In contrast, the second order shock capturing path-conservative Osher-type scheme presented here is more accurate and more efficient in the presence of strong shock waves and other discontinuities, since the DOT solver is a little dissipative and complete Riemann solver and since in the case of high Mach number flows, the flow velocity and the effective sound speed of the coupled fluid–structure system are rather close to each other. However, the main goal of this paper is not a detailed comparison with the method proposed in [20], but to provide an alternative discretization based on completely different design principles.

The rest of this paper is structured as follows: first, we present the set of equations that describes a compressible barotropic fluid flowing through a flexible visco-elastic tube with variable cross section. We assume an axially symmetric hydrostatic flow, where the fluid pressure is constant within each cross section. The governing PDE system is hyperbolic, with a conservative and a non-conservative part. In section 3 we present the Riemann solver that is used as a building block of the path-conservative finite volume scheme. In section 4 we provide some numerical results. The numerical scheme is applied to some classical Riemann problems for which a quasi-exact solution has been derived in [20,21]. We then simulate the dynamic behavior of a compressible fluid flowing through a flexible visco-elastic tube for which an exact solution exists in the frequency domain and for which also experimental data are available. The paper is rounded-off with some concluding remarks and an outlook to future research in section 5.

2. The governing PDE system of the coupled FSI problem

In the following section we introduce the PDE describing the flow of a barotropic fluid through a flexible tube, together with the visco-elastic behavior of the tube wall.

2.1. The cross-sectionally averaged Euler equations for the fluid

The axially symmetric flow of a compressible fluid through a tube with variable cross section $A(x,t)$ is modeled by the 1D cross-sectionally averaged Euler equations. We assume a barotropic fluid and a hydrostatic pressure that is constant within each cross section. The governing equations for the fluid read: