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An adaptive algorithm based on the shifted inverse iteration for the Steklov eigenvalue problem ☆

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ABSTRACT

This paper proposes and analyzes an a posteriori error estimator for the finite element multi-scale discretization approximation of the Steklov eigenvalue problem. Based on the a posteriori error estimates, an adaptive algorithm of shifted inverse iteration type is designed. Finally, numerical experiments comparing the performances of three kinds of different adaptive algorithms are provided, which illustrate the efficiency of the adaptive algorithm proposed here.

1. Introduction

In recent years, numerical methods for Steklov eigenvalue problems have attracted more and more scholars' attention (see, e.g., [3–6,9,13,21,26–28,31,36,40,41]). It is well known that in the numerical approximation of partial differential equations, the adaptive procedures based on a posteriori error estimates, due to the less computational cost and time, are the mainstream direction and have gained an enormous importance. The aim of this paper is to propose and analyze an a posteriori error estimator for the finite element multi-scale discretization approximation of the Steklov eigenvalue problem, based on which an adaptive algorithm is designed.

As for eigenvalue problems, till now, there are basically three ways to design adaptive algorithms as follows in which the a posterior error estimators are more or less the same but the equations solved in each iteration are different: I. Solve the original eigenvalue problem at each iteration. The convergence and optimality of this adaptive procedure have been studied in [18,19,21]. II. Inverse iteration type. [14,17,32,34,35,38] have studied and obtained the convergence of this method. III. Shifted inverse iteration type (see [23,30,42]). This paper studies the third type of adaptive method for the Steklov eigenvalue problem, and the special features are:

(1) As for the Steklov eigenvalue problem, so far, it has been discussed the first type of adaptive method of combining the a posteriori error estimates and adaptivity (see [21] or Algorithm 4.1 in this paper). As far as our information goes, there has not been any report on the other two kinds of adaptive methods. This paper designs the third type of adaptive algorithm based on the a posterior error estimates (see Algorithm 4.3 in this paper). Here we propose an a posteriori error estimator of residual type and give not only the global upper bound but also the local lower bound of the error which is important for the adaptive procedure.

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(2) [30] established and analyzed the a posteriori error estimates of the multi-scale discretization scheme for second order self-adjoint elliptic eigenvalue problems with homogeneous Dirichlet boundary value condition by means of the a posteriori error estimates of the associated boundary value problem and left the local lower bound of the error unproved, while this paper studies the a posteriori error estimates of the multi-scale discretization scheme by using the a posteriori error estimates of finite element eigenfunctions directly, and obtains the local lower bound of the error.

(3) Numerical experiments comparing the performances of three kinds of different adaptive methods mentioned above are provided. It can be seen from the numerical results that the adaptive algorithm of shifted inverse iteration type has advantages over the other two kinds. More precisely, comparing with the first type, to achieve the same accurate approximation, our method uses less computational time; and our adaptive algorithm can be used to seek efficiently approximations of any eigenpair of the Steklov eigenvalue problem, however, the algorithm of the second type (see Algorithm 4.2 in this paper) is only suitable to the smallest eigenvalue.

The rest of this paper is organized as follows. In section 2, some preliminaries needed in this paper are introduced. In section 3, a multi-scale discretization scheme is presented, and its a priori and a posteriori error estimates are given and analyzed, respectively. In section 4, three kinds of adaptive algorithms for the Steklov eigenvalue problem are presented, and finally numerical experiments are provided which illustrate the advantages of our algorithm.

2. Preliminaries

Let $H^t(\Omega)$ and $H^t(\partial\Omega)$ denote Sobolev spaces on Ω and $\partial\Omega$ with real order t, respectively. The norm in $H^t(\Omega)$ and $H^t(\partial\Omega)$ are denoted by $\|\cdot\|_t$ and $\|\cdot\|_{t,\partial\Omega}$, respectively. $H^0(\partial\Omega)=L_2(\partial\Omega)$.

In this paper, we will write $a \lesssim b$ to indicate that $a \leqslant Cb$ with $C>0$ being a constant depending on the data of the problem but independent of meshes generated by the adaptive algorithm.

We consider the following Steklov eigenvalue problem

$$-\Delta u + u = 0 \ \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = \lambda u \ \text{ on } \partial\Omega, \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain with θ being the largest inner angle of Ω and $\frac{\partial u}{\partial n}$ is the outward normal derivative.

The weak form of (2.1) is given by: find $\lambda \in \mathbb{R}$, $u \in H^1(\Omega)$ with $\|u\|_1 = 1$, such that

$$a(u,v) = \lambda b(u,v), \quad \forall v \in H^1(\Omega), \tag{2.2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v + uv dx, \ b(u,v) = \int_{\partial\Omega} uv ds,$$

$$\|u\|_b = b(u,u)^{\frac{1}{2}} = \|u\|_{0,\partial\Omega}.$$

It is easy to know that $a(\cdot,\cdot)$ is a symmetric, continuous and $H^1(\Omega)$-elliptic bilinear form on $H^1(\Omega)\times H^1(\Omega)$. So, we use $a(\cdot,\cdot)$ and $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)} = \|\cdot\|_1$ as the inner product and norm on $H^1(\Omega)$, respectively.

Let $\{\pi_h\}$ be a family of regular triangulations of Ω with the mesh diameter h, and $V_h \subset C(\overline{\Omega})$ be a space of piecewise linear polynomials defined on π_h.

The conforming finite element approximation of (2.2) is: find $\lambda_h \in \mathbb{R}$, $u_h \in V_h$ with $\|u_h\|_a = 1$, such that

$$a(u_h, v) = \lambda_h b(u_h, v), \quad \forall v \in V_h. \tag{2.3}$$

Consider the following source problem (2.4) associated with (2.2) and the approximate source problem (2.5) associated with (2.3), respectively.

Find $w \in H^1(\Omega)$, such that

$$a(w,v) = b(f,v), \quad \forall v \in H^1(\Omega). \tag{2.4}$$

Find $w_h \in V_h$, such that

$$a(w_h, v) = b(f,v), \quad \forall v \in V_h. \tag{2.5}$$

From [20] we know that the following regularity estimates hold for (2.4).

Lemma 2.1. *If $f \in L_2(\partial\Omega)$, then there exists a unique solution $w \in H^{1+\frac{r}{2}}(\Omega)$ to (2.4), and*

$$\|w\|_{1+\frac{r}{2}} \lesssim \|f\|_{0,\partial\Omega}; \tag{2.6}$$

if $f \in H^{\frac{1}{2}}(\partial\Omega)$, then there exists a unique solution $w \in H^{1+r}(\Omega)$ to (2.4), and