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On a local Fourier analysis for overlapping block smoothers on triangular grids

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ABSTRACT

A general local Fourier analysis for overlapping block smoothers on triangular grids is presented. This analysis is explained in a general form for its application to problems with different discretizations. This tool is demonstrated for two different problems: a stabilized linear finite element discretization of Stokes equations and an edge-based discretization of the curl–curl operator by lowest-order Nédélec finite element method. In this latter, special Fourier modes have to be considered in order to perform the analysis. Numerical results comparing two- and three-grid convergence factors predicted by the local Fourier analysis to real asymptotic convergence factors are presented to confirm the predictions of the analysis and show their usefulness.

1. Introduction

As is well-known, multigrid methods [3,11] are among the most powerful techniques for the efficient resolution of the large systems of equations arising from the discretization of partial differential equations. Since the 70's, when these methods were developed, they have become very popular among the scientific community. They have the nice property of requiring a computational work of the order of the number of unknowns of the problem, at least for elliptic problems. Besides, they have also been applied to more complicated problems, for example see [30], providing very good results.

The efficiency and the robustness of a multigrid method is essentially influenced by the smoothing algorithm. We want to study the class of multiplicative Schwarz smoothers. Basically, they can be described as an overlapping block Gauss–Seidel method, where a small linear system of equations for each grid point has to be solved in each smoothing step. This type of smoothers is characterized by its ability to deal with saddle point problems and equations where the terms grad–div or curl–curl dominate. A particular case of such relaxation is the so-called Vanka smoother, introduced in [32] for solving the staggered finite difference discretization of the Navier–Stokes equations.

Local Fourier analysis (LFA, or local mode analysis) is a commonly used approach for analyzing the convergence properties of geometric multigrid methods. In this analysis an infinite regular grid is considered and boundary conditions are not taken into account. LFA was introduced by Brandt in [3] and afterward extended in [4]. A good introduction can be found in the paper by Stüben and Trottenberg [29] and in the books by Wesseling [33], Trottenberg et al. [30], and Wienands

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and Joppich [34]. LFA was generalized to triangular grids in [7], for discretizations based on linear finite element methods. Afterwards, this generalization has been extended to systems of partial differential equations [8,9] and to high-order finite element discretizations [25].

To our knowledge, there are only few papers dealing with local Fourier analysis for overlapping smoothers, all of them for discretizations on rectangular grids. This analysis was performed in [28] for the staggered finite-difference discretizations of the Stokes equations, and in [22] for a mixed finite element discretization of the Laplace equation. In [2], an LFA to analyze an additive Schwarz smoother for a curl–curl model problem is proposed. A multicolored version was considered in the way that the corresponding analysis does not consider the special techniques to study the standard overlapping smoothers. In [24] an LFA for overlapping block smoothers on triangular grids is presented. This tool was applied to linear finite element discretizations for poroelasticity problems. Later, in [20], the analysis for such overlapping block smoothers is performed on rectangular grids for finite element discretizations of the grad–div, curl–curl and Stokes equations. Here, we present and extend this analysis to general discretizations on triangular grids, including some special techniques for the case of edge-based discretizations. Two model problems are chosen to show this analysis, but we keep in mind that it can be carried over to a variety of other problems and other overlapping smoothers. The considered problems are the discretization by stabilized linear finite elements of the Stokes problem, and the low-order Nédélec's edge elements for the curl–curl equation. Regarding the Stokes system, we perform an exhaustive two-grid local Fourier analysis for the full- and diagonal (much cheaper) versions of the overlapping block smoother. Apart from this, also a three-grid analysis is developed in order to obtain more insight. This analysis has to deal with the difficulties inherent to the treatment of this kind of smoothers and also it has to take into account the corresponding extension of LFA to triangular grids. Apart from these difficulties, for the second model problem we need to extend this analysis to edge-based discretizations on which different stencils appear depending on the type of edge. This makes necessary the introduction of special Fourier modes in order to perform the analysis. Also for this test we perform a two- and three-grid LFA to analyze the differences in the performance of W- and V-cycles.

The structure of the paper is as follows. In Section 2, a general description of the class of overlapping block smoothers is done, together with the development of a suitable local Fourier analysis for this type of relaxation procedures. Two- and three-grid local Fourier analysis are performed. Sections 3 and 4 are focused on a detailed description of the local Fourier analysis of two particular overlapping block smoothers for the solution of two different model problems. More concretely, in Section 3 a stabilized linear finite element discretization of the Stokes equations is considered and in Section 4 we deal with an edge-based discretization of a curl–curl problem by using low order Nédélec finite elements. Finally, in Section 5 some conclusions are drawn.

2. Local Fourier analysis for overlapping block smoothers

2.1. Description of the smoother

Point-wise iterative methods can be generalized to block-wise iterative schemes by updating a set of unknowns at each time, instead of only one. To this end, the grid is split into blocks and the equations corresponding to the grid-points in each block are simultaneously solved as a system of equations. Block-wise schemes become very attractive when anisotropies appear, especially when they are combined with a problem-dependent ordering of the blocks, since point-wise relaxation techniques lose their smoothing property. Many arbitrary splittings of the mesh can be considered. For example, it is possible to allow the blocks to overlap, what gives rise to the class of overlapping block iterations, where smaller local problems are solved and combined via a multiplicative Schwarz method. They were introduced by Vanka in [32], and in [27] a theoretical basis for this approach was provided. These smoothers, also known as coupled or box-relaxation, consist of decomposing the mesh into small subdomains and treating them separately. Therefore, one relaxation step consists of a loop over all subdomains, solving for each one the system arising from the corresponding equations. Next, we give a more detailed description of the iterative method. We consider a linear system of equations $A_h u_h = f_h$, which, in our case, arises from the discretization of a PDE problem. Vector u_h is composed of unknowns corresponding to m different variables. More concretely, N_i unknowns of each variable i are considered. Let B be the subset of unknowns involved in an arbitrary block, that is, $B = \{u^1_{k_1(1)}, \ldots, u^1_{k_1(n_1)}, \ldots, u^m_{k_m(1)}, \ldots, u^m_{k_m(n_m)}\}$, where $k_i(1), \ldots, k_i(n_i)$ are the global indexes of the n_i unknowns corresponding to variable i. In order to obtain the matrix A^B_h of the system to solve associated with block B, we introduce the matrix V_B representing the projection operator from the vector of all unknowns to the vector of the unknowns involved in the block, as the following block-diagonal matrix

$$V_B = \begin{pmatrix} V^1_B & & \\ & \ddots & \\ & & V^m_B \end{pmatrix}. \tag{1}$$

Here, each block V^i_B is a $(n_i \times N_i)$-matrix, whose jth-row is the $k_i(j)$th-row of the identity matrix of order N_i. In this way, matrix A^B_h can be defined as

$$A^B_h = V_B A_h V^T_B. \tag{2}$$