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Series expansion method for weakly singular Volterra integral equations ☆

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ABSTRACT

We propose series expansion method to solve VIEs (Volterra integral equations) with smooth given functions, including weakly singular VIEs possessing unsmooth solution. The key step in proposed method is to approximate given functions by their own Chebyshev Gauss–Lobatto interpolation polynomials. Lubich's results play an important role in the proposed method for weakly singular VIEs. Convergence analysis is provided for proposed method. Numerical experiments are carried out to confirm theoretical results.

1. Introduction

Volterra integral equations (VIEs) arise in many problems such as mathematical problems of radiation equilibrium, the particle transport problems of astrophysics and reactor theory, and radiation heat transfer problems. There exist numerical methods for solving VIEs such as cubic spline approximation [20], Runge–Kutta methods [8], Haar wavelet method [21], Legendre wavelet method [18], optimal control method [9], quadrature rule method [1], piecewise polynomial collocation method [2]. Tang and his coworkers [26] proposed Legendre spectral collocation method to solve VIEs. Subsequently, Chen and Tang [5–7] proposed Jacobi spectral collocation method to solve weakly singular VIEs. In [14], Li, Tang and Xu proposed a parallel spectral method to solve VIEs. In their excellent book [23], Shen and his coworkers Tang and Wang provided a detailed presentation of basic spectral algorithms, a systematical presentation of basic convergence theory and error analysis for spectral methods, and some illustrative applications of spectral methods. Referring to above literatures, Chen and Gu [4, 10–13] proposed piecewise Legendre spectral collocation method and Chebyshev spectral collocation method for Volterra type integral equations.

Series expansion methods are classic numerical methods for differential or integral equations. Sun and Zhang [24,25] investigated series expansion methods for a class of second order differential equations in complex. In [16,17,21,22], Sezer (1994), Maleknejad and Aghazadeh (2005), Maleknejad, Sohrabi and Rostami (2007), Rabbani, Maleknejad and Aghazadeh (2007) proposed series expansion method to solve VIEs or VIEs system.

In this paper series expansion method is used to solve VIE

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$$y(t) = g(t) + \int_0^t K(t,s)y(s)ds,\ t \in [0,1], \tag{1}$$

and weakly singular VIE

$$\widehat{y}(t) = g(t) + \int_0^t (t-s)^{-\mu} K(t,s)\widehat{y}(s)ds,\ t \in [0,1],\ 0 < \mu < 1, \tag{2}$$

where $y(t)$ and $\widehat{y}(t)$ are unknown functions defined on $[0,1]$, given functions $g(t)$, $K(t,s)$ possess continuous derivatives of at least m order in their own definition. For VIEs defined on $[0,T]$, we can use a variable transformation changing them to the new ones of the form (1) or (2). The solution of (1) inherits the regularity of $g(t)$ and $K(t,s)$ (see [2]). In general cases, the solution of (2) is not a smooth function if given functions $g(t)$, $K(t,s)$ are smooth on their own definition domains (see [2]). The derivative of $y(t)$ behaves like $t^{-\mu}$. This provides difficulty for solving numerically these equations. Lubich (1983) [15] discovered that there exists a function of two variables $Y(z_1,z_2)$ analytic near $(0,0)$ such that $\widehat{y}(t) = Y(t, t^{1-\mu})$.

In our method, we approximate $g(t)$ and $K(t,s)$ by their own Chebyshev Gauss–Lobatto interpolation polynomial of N degree, i.e., $g^N(t)$ and $K^N(t,s)$. Then (1) and (2) can be approximated respectively by

$$y^N(t) = g^N(t) + \int_0^t K^N(t,s)y^N(s)ds \tag{3}$$

and

$$Y^N(t,t^{1-\mu}) = g^N(t) + \int_0^t (t-s)^{-\mu} K^N(t,s) Y^N(s,s^{1-\mu})ds, \tag{4}$$

where $y^N(t)$, $Y^N(t,t^{1-\mu})$ are approximations to $y(t)$ and $Y(t,t^{1-\mu})$ respectively. In order to use the series expansion method, we write $y^N(t)$, $Y^N(t,t^{1-\mu})$, $g^N(t)$ and $K^N(t,s)$ in power series form,

$$y^N(t) = \sum_{h=0}^{\infty} y_h t^h, \quad Y^N(t,t^{1-\mu}) = \sum_{i\geq 0, j\geq 0} Y_{ij} t^i (t^{1-\mu})^j, \tag{5}$$

$$g^N(t) = \sum_{i=0}^{N} g_i t^i, \quad K^N(t,s) = \sum_{0\leq i,j\leq N} k_{ij} t^i s^j. \tag{6}$$

Comparing coefficients of t^h and $t^i(t^{1-\mu})^j$ at both sides of (3) and (4) respectively, we can get the relation between y_h, Y_{ij} and g_i, k_{ij}. Finally recursion formulas of y_h and Y_{ij} are obtained. Convergence analysis is provided to show that errors decay at rate $N^{(1/2)-m}$ in infinite norm which shows that more collocation points (N is larger) employed to approximate $g(t)$ and $K(t,s)$, and more regularity $g(t)$ and $K(t,s)$ possessed (m is larger), will enhance the precision of the numerical solutions. Numerical examples are provided to confirm these theoretical results.

This paper is organized as followings. In Sec. 2, we present series expansion method for VIEs (1) and weakly singular VIEs (2). Some useful lemmas for convergence analysis are given in Sec. 3. The convergence analysis for proposed methods is provided in Sec. 4. Numerical experiments are carried out in Sec. 5. Finally, in Sec. 6, we end with conclusion and future work.

2. Series expansion method

In this section, we present series expansion method for VIE (1) and weakly singular VIE (2).

Let $\{t_i : i = 0, 1, \cdots, N\}$ be Chebyshev Gauss–Lobatto points in interval $[0,1]$, i.e.,

$$t_i := \frac{1}{2}(x_i + 1),\ i = 0, 1, \cdots, N,$$

where $\{x_i : i = 0, 1, \cdots, N\}$ are Chebyshev Gauss–Lobatto points in $[-1,1]$. The j-th Lagrange interpolation basic function associated with $\{t_i : i = 0, 1, \cdots, N\}$ is

$$L_j(t) := \prod_{i=0, i\neq j}^{N} (t - t_i)/(t_j - t_i). \tag{7}$$

We use