Applied Numerical Mathematics 105 (2016) 124–147

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Unilateral problem for the Stokes equations: The well-posedness and finite element approximation

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A R T I C L E   I N F O

Article history:
Received 17 July 2015
Received in revised form 15 December 2015
Accepted 10 March 2016
Available online 1 April 2016

Keywords:
Stokes equations
Finite element approximation
Unilateral boundary condition

A B S T R A C T

We consider the stationary Stokes equations under a unilateral boundary condition of Signorini's type, which is one of artificial boundary conditions in flow problems. Well-posedness is discussed through its variational inequality formulation. We also consider the finite element approximation for a regularized penalty problem. The well-posedness, stability and error estimates of optimal order are established. The lack of a coupled Babuška and Brezzi's condition makes analysis difficult. We offer a new method of analysis. Particularly, our device to treat the pressure is novel and of some interest. Numerical examples are presented to validate our theoretical results.

1. Introduction

We suppose that Ω is a bounded domain in $\mathbb{R}^d$ with $d = 2, 3$ and that the boundary $\partial\Omega$ is comprised of three parts S_1, S_2 and Γ. Those S_1, S_2 and Γ are assumed to be smooth but the whole boundary $\partial\Omega$ is not necessarily smooth. One might imagine a branched pipe resembling that depicted in Fig. 1. The first purpose of this paper is to study the well-posedness of the following unilateral boundary value problem for the Stokes equations

$$-\nu\Delta u + \nabla p = f, \quad \nabla\cdot u = 0 \qquad \text{in } \Omega, \tag{1a}$$

$$u = 0 \qquad \text{on } S_1 \cup S_2, \tag{1b}$$

$$u_n + g_n \geq 0, \qquad \text{on } \Gamma, \tag{1c}$$

$$\tau_n(u, p) + \alpha_n \geq 0 \qquad \text{on } \Gamma, \tag{1d}$$

$$(u_n + g_n)(\tau_n(u, p) + \alpha_n) = 0 \qquad \text{on } \Gamma, \tag{1e}$$

$$\tau_T(u) + \alpha_T = 0 \qquad \text{on } \Gamma \tag{1f}$$

for velocity $u = (u_1, \dots, u_d)$ and pressure p with density $\rho = 1$ and kinematic viscosity ν of the viscous incompressible fluid under consideration. Therein,

$$\tau(u, p) = \sigma(u, p)n$$

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http://dx.doi.org/10.1016/j.apnum.2016.03.002

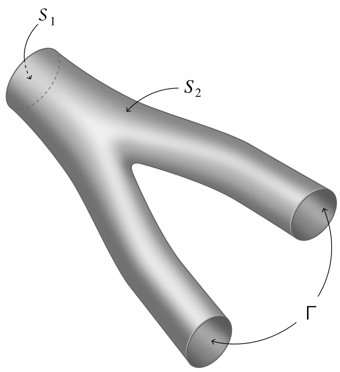

Fig. 1. Example of Ω (branched pipe).

denotes the traction vector on $\partial\Omega$, where n is the outward normal vector to $\partial\Omega$, $\sigma(u,p) = (\sigma_{ij}(u,p))_{1\le i,j\le d} = -pI + 2\nu D(u)$ the stress tensor, $D(u) = (D_{ij}(u))_{1\le i,j\le d} = \frac{1}{2}\left(\nabla u + \nabla u^{\mathrm{T}}\right)$ the deformation-rate tensor and I the identity matrix. For a vector-valued function v on $\partial\Omega$, its normal and tangential components are denoted, respectively, as

$$v_n = v \cdot n, \quad v_T = v - v_n n.$$

Particularly, $\tau_n(u,p) = \tau(u,p)\cdot n$ and $\tau_T(u) = \tau(u,p) - \tau_n(u,p)n$ respectively denote normal and tangential traction vectors. Moreover, f, g and α are prescribed functions. We also consider the finite element approximation for a regularized penalty problem to (1) which is given as

$$-\nu\Delta u + \nabla p = f, \quad \nabla\cdot u = 0 \qquad \text{in } \Omega, \tag{2a}$$

$$u = 0 \qquad \text{on } S_1 \cup S_2, \tag{2b}$$

$$\tau_n(u,p) + \alpha_n = \frac{1}{\varepsilon}\phi_\delta(u_n + g_n) \qquad \text{on } \Gamma, \tag{2c}$$

$$\tau_T(u) + \alpha_T = 0 \qquad \text{on } \Gamma \tag{2d}$$

with $0 < \varepsilon \ll 1$ and $0 < \delta \ll 1$. Therein, $\phi_\delta(s)$ is a C^1 regularization of $[s]_- = \max\{0, -s\}$. We can take, for example,

$$\phi_\delta(s) = \begin{cases} 0 & (s \ge 0) \\ (\sqrt{s^2+\delta^2} - \delta) & (s < 0). \end{cases} \tag{3}$$

First, we explain our motivation for studying (1) and (2). In numerical simulation of real-world flow problems, we often encounter some issues related to artificial boundary conditions. A typical and important example is the blood flow problem in the large arteries, where the blood is assumed to be a viscous incompressible fluid (see [17,32]). The blood vessel is modeled as a branched pipe as illustrated, for example, in Fig. 1. Then, for $T > 0$, we consider the Navier–Stokes equations for velocity $v = (v_1, \ldots, v_d)$ and pressure q,

$$v_t + (v\cdot\nabla)v = \nabla\cdot\sigma(v,q) + f, \quad \nabla\cdot v = 0 \qquad \text{in } \Omega\times(0,T), \tag{4a}$$

$$v = b \qquad \text{on } S_1\times(0,T), \tag{4b}$$

$$v = 0 \qquad \text{on } S_2\times(0,T) \tag{4c}$$

with the initial condition $v|_{t=0} = v_0$. We are able to give a velocity profile $b = b(x,t)$ at the *inflow boundary* S_1. The flow is presumed to be a perfect non-slip flow on the wall S_2. Then, the blood flow simulation is highly dependent on the choice of artificial boundary conditions posed on the *outflow boundary* Γ.

An earlier paper by Zhou and Saito [34] presented discussion of the free-traction condition

$$\tau(v,q) = 0 \quad \text{on } \Gamma, \tag{5}$$

which is one of the common outflow boundary conditions (see [19,22]), and some nonlinear energy-preserving boundary conditions (see [3,7,8,11,12]) from the view-point of energy inequality. Moreover, we proposed a new outflow boundary condition as

$$v_n \ge 0, \quad \tau_n(v,q) \ge 0, \quad v_n\tau_n(v,q) = 0, \quad \tau_T(v) = 0 \quad \text{on } \Gamma. \tag{6}$$

This is an analogy to Signorini's condition in the theory of elasticity (see [24]). It is indeed a generalization of the free-traction condition (5), as