Applied Numerical Mathematics 92 (2015) 1–20

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Convergence of two-dimensional staggered central schemes on unstructured triangular grids ☆

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ARTICLE INFO

Article history:
Received 10 June 2013
Received in revised form 21 December 2014
Accepted 5 January 2015
Available online 22 January 2015

Keywords:
Central schemes
Unstructured grids
Convergence analysis
Stability condition

ABSTRACT

In this paper, we present a convergence analysis of a two-dimensional central finite volume scheme on unstructured triangular grids for hyperbolic systems of conservation laws. More precisely, we show that the solution obtained by the numerical base scheme presents, under an appropriate CFL condition, an optimal convergence to the unique entropy solution of the Cauchy problem.

1. Introduction

Very few are the convergence studies of central finite volume schemes on staggered unstructured grids for multi-dimensional domains. In [4,6,8] convergence on a fixed grid has been proven. Whereas in [2] and [1] a proof was developed for the case of linear conservation laws on barycentric two dimensional grids. Convergence of the first order Lax–Friedrichs scheme on the same special staggered grids for nonlinear scalar problems was provided in [7]. Assuming a discontinuous initial function $u_0 \in L^\infty(\mathbb{R}^d) \cap BV_{loc}(\mathbb{R}^d)$ for resolving the nonlinear hyperbolic equation with the Lax–Friedrichs type finite volume scheme on unstructured grids, Küther [9] proved that an error estimate of order 0.25 is expected, which is not optimal. On the other hand, Sabac [10] demonstrated that $h^{\frac{1}{2}}$ is optimal for first order schemes.

In this work, we present a convergence analysis of the two-dimensional central finite volume schemes on unstructured triangular grids recently developed in [11] for hyperbolic systems of conservation laws. These schemes are particularly interesting when solving hyperbolic systems on irregular geometries since they use triangular cells as control volumes, which leads to lower computational costs and faster simulations as compared to other finite volume methods. Furthermore, the choice of the dual control cells of the finite volume method developed in [11] (quadrilateral cells) lead to a an easier and simpler construction of the numerical scheme as compared to the case of finite volume methods with barycentric dual cells such as in [3]

The proposed analysis is based on the convergence proof for the barycentric-cells central finite volume method for nonlinear hyperbolic equations established in [7]; this proof was developed for central finite volume schemes on barycentic control cells defined over triangular meshes with quadrilateral staggered dual cells.

☆ This paper is dedicated to the memory of professor Paul Arminjon (1941–2011).

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http://dx.doi.org/10.1016/j.apnum.2015.01.003

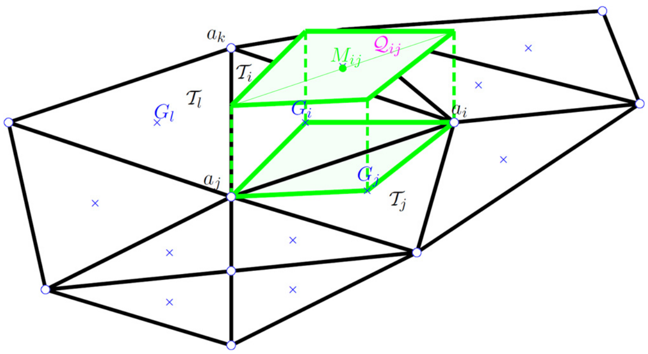

Fig. 1. Triangular control cells of the original grid.

In our convergence analysis we adopt the convergence study technique developed in the work of Haasdonk, Kröner and Rohde [7] and develop a convergence analysis of the central finite volume schemes on non-barycentric triangular cells [11] in the case of scalar nonlinear hyperbolic conservation laws. More precisely, we show that evolving a piecewise constant numerical solution using the Lax–Friedrichs version of the central finite volume method on unstructured triangular cells [11] leads to a first-order convergence rate. Numerical experiments validate the expected order of convergence.

2. A Lax–Friedrichs scheme extension on unstructured triangular grids

In this section we briefly describe the Lax–Friedrichs extension to the case of two-dimensional unstructured triangular grids. The convergence study of this extension will be presented in Section 3.

Starting with the two-dimensional conservation law

$$\begin{cases} u_t + \nabla \cdot \mathcal{F}(u) = 0, & (x, y) \in \Omega, \quad t > 0, \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega \end{cases} \tag{1}$$

where u is a scalar field, $u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\mathcal{F} \in C^1(\mathbb{R}^2) : \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^2$ is such that $((x, y), t, u(x, y, t)) \mapsto \mathcal{F}((x, y), t, u(x, y, t))$ and $\frac{\partial \mathcal{F}}{\partial u}$ is locally Lipschitz continuous. We denote by f and $g \in C^1(\mathbb{R})$ the components of the vector $\mathcal{F}$.

We assume that the computational domain Ω is discretized using a finite element triangulation T_h, and we assume that the initial condition is defined at the centroids $G_i = (x_i, y_i)$ of the triangles T_i of the mesh. The method we present evolves a piecewise constant numerical solution on two staggered grids at consecutive time-steps. Assuming that u_i^n approximates the solution $u(x_i, y_i, t^n)$ at time $t = t^n$ on the control cells T_i, the solution at the next time-step $t^{n+1} = t^n + \Delta t$ is computed on the staggered cells Q_{ij} obtained by joining the centroids G_i and G_j of two adjacent triangular cells T_i and T_j to the endpoints a_i and a_j of their common edge $[a_i, a_j]$ as shown in Fig. 1. The solution at time t^{n+2} will be calculated at the centers G_i of the triangles T_i. Alternating the numerical solution at consecutive time-steps on the original and staggered grids avoids the time consuming process of solving the Riemann problems arising at the cell interfaces.

Assuming that u_i^n is known at time t^n on the cells T_i, the solution u_{ij}^{n+1} at time t^{n+1} is calculated as follows.

We first integrate the conservation law (1) on the domain $Q_{ij} \times [t^n, t^{n+1}]$ and we apply Greens formula, we obtain:

$$\int_{Q_{ij}} u(x, y, t^{n+1}) d\mathcal{A} - \int_{Q_{ij}} u(x, y, t^n) d\mathcal{A} + \int_{t^n}^{t^{n+1}} \int_{\partial Q_{ij}} \big(f\big(u(x, y, t)\big)\nu_x + g\big(u(x, y, t)\big)\nu_y\big) d\sigma\, dt = 0, \tag{2}$$

where (ν_x, ν_y) denotes the unit outward normal vector to the boundary ∂Q_{ij} of the quadrilateral dual cell Q_{ij} (see Fig. 2). Since the numerical solution at any time t^n is piecewise constant defined on the control cells, and knowing that $Q_{ij} = (Q_{ij} \cap T_i) \cup (Q_{ij} \cap T_j)$, we obtain

$$u_{ij}^{n+1} = \frac{1}{\mathcal{A}(Q_{ij})} \big(u_i^n \mathcal{A}(Q_{ij} \cap T_i) + u_j^n \mathcal{A}(Q_{ij} \cap T_j)\big) - \frac{1}{\mathcal{A}(Q_{ij})} \int_{t^n}^{t^{n+1}} \int_{\partial Q_{ij}} \big(f\big(u(x, y, t)\big)\nu_x + g\big(u(x, y, t)\big)\nu_y\big) d\sigma\, dt, \tag{3}$$

where $\mathcal{A}(Q_{ij})$ denotes the area of the quadrilateral cell Q_{ij}. The flux integrals in time and space are estimated using first-order quadrature rules as follows: