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A linearised singularly perturbed convection–diffusion problem with an interior layer

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ABSTRACT

A linear time dependent singularly perturbed convection–diffusion problem is examined. The convective coefficient contains an interior layer (with a hyperbolic tangent profile), which in turn induces an interior layer in the solution. A numerical method consisting of a monotone finite difference operator and a piecewise-uniform Shishkin mesh is constructed and analysed. Neglecting logarithmic factors, first order parameter uniform convergence is established.

1. Introduction

To construct layer adapted meshes (such as the piecewise-uniform Shishkin mesh [2]) for a class of singularly perturbed problems, whose solutions contain boundary layers, it is necessary to identify both the location and the width of any boundary layers present in the solution. In addition to boundary layers, interior layers can also appear in the solutions of singularly perturbed problems. In the context of time dependent problems, an additional issue with interior layers is that the location of the layer can move with time. Here we focus on parabolic problems with interior layers, whose location is approximately known at all time.

Consider singularly perturbed parabolic problems of convection–diffusion type, which take the form: Find u such that

$$-\varepsilon u_{xx} + au_x + bu + cu_t = f, \quad (x,t) \in (0,1) \times (0,T], \quad b \geq 0, c > 0; \tag{1a}$$

$$0 < \varepsilon \ll 1, \qquad u(0,t), u(1,t), u(x,0) \quad \text{specified.} \tag{1b}$$

In [1,10], interior layers appeared in the solution of (1), in the special case where the convective coefficient $a(x)$ was assumed to be discontinuous across a curve $\Gamma_1 := \{(d(t),t) | t \in [0,T], 0 < d(t) < 1\}$ and to have the particular sign pattern $a(x) > 0, x < d(t); a(x) < 0, x > d(t)$. In [10], by mapping this curve Γ_1 to the vertical line $x = d(0)$, a piecewise-uniform Shishkin mesh [2] was constructed to align the fine mesh with this curve. This mesh enabled a parameter-uniform numerical method [2] for problem (1) to be constructed. In [3], interior layers appeared in the solution of (1), in the case where the initial condition $u(x,0)$, contained it's own interior layer. In the case of [3], the convective coefficient $a(t)$ was assumed to be space independent, smooth and of one sign. The reduced initial condition (set $\varepsilon = 0$) was discontinuous at some point

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$x = d$ and this discontinuity was transported along the characteristic curve $\Gamma_2 := \{(d(t), t) | t \in [0, T], d'(t) = a(t), d(0) = d\}$, associated with the reduced hyperbolic problem $av_x + bv + cv_t = f$. Again, a parameter-uniform numerical method (akin to the method analysed in [1]) was shown [3] to be (essentially) first order uniformly convergent. In the current paper, an interior layer appears in the solution of (1) due to the fact that the convective coefficient $a_\varepsilon(x, t)$ is assumed to be smooth, but to rapidly change from positive to negative values within the domain. In the limiting case of $\varepsilon = 0$, the convective coefficient of the reduced differential equation will be discontinuous. This problem may be viewed as a time dependent version of the ordinary differential equation examined in [9].

Under certain conditions [11] the solution of the quasilinear problem

$$-\varepsilon y_{xx} + yy_x + by + y_t = 0, \quad x \in (0, 1), t > 0, \quad b \geq 0; \tag{2a}$$

$$y(0, t) > 0, \; y(1, t) < 0, \quad y(x, 0) \text{ specified}; \tag{2b}$$

will exhibit an interior layer [4] centred along some curve $\Gamma^* := \{(q(t), t), t > 0\}$, which has a hyperbolic tangent profile. In the case of the corresponding Cauchy problem posed on the unbounded domain $(x, t) \in (-\infty, \infty) \times (0, \infty)$ with a smooth initial condition $y(x, 0) = g(x), x \in (-\infty, \infty)$, there will be an initial phase before the interior layer is fully formed [5]. After this initial phase, the solution always exhibits a sharp interior layer and the location of the centre of this layer will vary with time. Our interest is in studying numerical methods that will track the solution, after the formative phase has elapsed. Hence, we wish to consider the behaviour of the solution of the boundary/initial value problem (2), when the initial condition already contains an interior layer.

The location of this curve Γ^* (across which the reduced solution is discontinuous) can be estimated using asymptotic expansions [7,11]. To the left of Γ^*, the solution can be viewed as being the sum of two components v_L, w_L, where the regular component v_L is composed of an asymptotic expansion of the form $v_L = v_0^L + \varepsilon v_1^L + \varepsilon^2 v_2^L + ..$; and v_0^L satisfies the reduced nonlinear first order differential equation (set $\varepsilon = 0$) and $v_L(0, t) = y(0, t), v_L(x, 0) = y(x, 0)$. The regular components v_L, v_R are constructed so that v_L (v_R) satisfies the quasilinear differential equation when $x < d(t)$ ($x > d(t)$) and their partial derivatives (up to a certain order) are bounded independently of ε. However, in general, $v_L(d(t), t) \neq v_R(d(t), t)$. To the left of Γ^*, the decomposition is also designed so that the singular component w_L satisfies bounds [11] of the following form

$$\left| \frac{\partial^{i+j} w_L(x, t)}{\partial x^i \partial t^j} \right| \leq C\varepsilon^{-i} e^{-\theta(d(t)-x)/\varepsilon}, \quad \theta > 0; \quad x < d(t), t > 0.$$

In this paper, we formulate a linearised version of the above quasilinear problem (2). The definition of the linearised problem is motivated by the above decomposition of the solution into regular and singular components.

In Section 2 we state the continuous problem (3) examined in this paper and impose constraints (4) on the convective coefficient a that mimic that character of the continuous solution itself. These assumptions on a confine the location of the interior layer to an $O(\varepsilon)$ neighbourhood of its initial location. The continuous solution is decomposed into the sum of a discontinuous regular component and a discontinuous interior layer component. Pointwise bounds on both components and on their derivatives are established. In Section 3, based on the bounds established on the layer component, a piecewise-uniform Shishkin mesh is constructed. In Section 4, the numerical approximations, generated from using a simple finite difference operator on this layer-adapted mesh, are shown to converge ε-uniformly. Some numerical results are presented and discussed in the final section.

Notation: Throughout this paper C denotes a generic constant which is independent of ε and all mesh parameters. Also $\|\cdot\|$ denotes the pointwise maximum norm, which will be subscripted when the norm is restricted to a subdomain. The space $C^{0+\gamma}(D)$ is the set of all functions that are Hölder continuous of degree γ with respect to the metric $\|\cdot\|_p$, where for all $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$

$$\|\mathbf{u} - \mathbf{v}\|_p^2 := (u_1 - v_1)^2 + |u_2 - v_2|.$$

For f to be in $C^{0+\gamma}(D)$ then $f \in C^0(D)$ and the following semi-norm needs to be finite

$$\lceil f \rceil_{0+\gamma, D} := \sup_{\mathbf{u} \neq \mathbf{v}, \; \mathbf{u}, \mathbf{v} \in \mathbf{D}} \frac{|f(\mathbf{u}) - f(\mathbf{v})|}{\|\mathbf{u} - \mathbf{v}\|_p^\gamma}.$$

The space $C^{n+\gamma}(D)$ is the set of all functions, whose derivatives of order n are Hölder continuous of degree $\gamma > 0$ in the domain D. That is,

$$C^{n+\gamma}(D) := \{z : \frac{\partial^{i+j} z}{\partial x^i \partial t^j} \in C^\gamma(D), \; 0 \leq i + 2j \leq n\}.$$

Also $\|\cdot\|_{n+\gamma}$ and $\lceil \cdot \rceil_{n+\gamma}$ are the associated Hölder norms and semi-norms defined by

$$\|v\|_{n+\gamma} := \sum_{0 \leq k \leq n} |v|_k + \lceil v \rceil_{n+\gamma}, \; |v|_k := \sum_{k=i+2j} \left\| \frac{\partial^{i+j} v}{\partial x^i \partial t^j} \right\|, \; \lceil v \rceil_{n+\gamma} := \sum_{i+2j=n} \left\lceil \frac{\partial^{i+j} v}{\partial x^i \partial y^j} \right\rceil_{0+\gamma}.$$