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Relative perturbation theory for definite matrix pairs and hyperbolic eigenvalue problem

Ninoslav Truhar, Suzana Miodragović *

Department of Mathematics, J. J. Strossmayer University of Osijek, Trg Ljudevita Gaja 6, HR 31000 Osijek, Croatia

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In this paper, new relative perturbation bounds for the eigenvalues as well as for the eigensubspaces are developed for definite Hermitian matrix pairs and the quadratic hyperbolic eigenvalue problem. First, we derive relative perturbation bounds for the eigenvalues and the $\sin\Theta$ type theorems for the eigensubspaces of the definite matrix pairs (A, B), where both $A, B \in \mathbb{C}^{m\times m}$ are Hermitian nonsingular matrices with particular emphasis, where B is a diagonal of ± 1. Further, we consider the following quadratic hyperbolic eigenvalue problem $(\mu^2 M + \mu C + K)v = 0$, where $M, C, K \in \mathbb{C}^{n\times n}$ are given Hermitian matrices. Using proper linearization and new relative perturbation bounds for definite matrix pairs (A, B), we develop corresponding relative perturbation bounds for the eigenvalues and the $\sin\Theta$ type theorems for the eigensubspaces for the considered quadratic hyperbolic eigenvalue problem. The new bounds are uniform and depend only on matrices M, C, K, perturbations δM, δC and δK and standard relative gaps. The quality of new bounds is illustrated through numerical examples.

1. Introduction

In this paper, we present relative perturbation theory for the definite Hermitian matrix pairs (A, B), where both A and B are indefinite nonsingular Hermitian matrices for which there exists $\alpha \in \mathbb{R}$ such that $A - \alpha B$ is a positive definite matrix. One of the ways in which α can be chosen is given by Veselić [26, Theorem A1].

Moreover, by using these results we will develop the relative perturbation bounds for a relative change in the eigenvalues, as well as the $\sin\Theta$ type theorems for the corresponding eigensubspaces, for the so-called hyperbolic eigenvalue problem (HEP). The HEP is defined as

$$(\mu^2 M + \mu C + K)v = 0\,,$$

where $M, C, K \in \mathbb{C}^{n\times n}$ are given Hermitian matrices of order n, such that all eigenvalues μ_i, $i = 1, \dots, 2n$ are real and lie in the left (or right)-half plane. Our approach uses a linearization of the HEP and then a simple application of the bounds obtained for the definite Hermitian matrix pairs (A, B).

The main contributions of the paper are relative perturbation bounds for the HEP, which depend only on the matrices M, C and K (more precisely, our bounds depend on terms which involve L_M, L_K and C, where $M = L_M L_M^*$ and $K = L_K L_K^*$), corresponding perturbations δM, δC and δK and standard relative gaps.

* Corresponding author.
E-mail addresses: ntruhar@mathos.hr (N. Truhar), ssusic@mathos.hr (S. Miodragović).

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These kinds of bounds provide estimations for relative eigenvalue perturbations only by using L_M, L_K and C (where $M = L_M L_M^*$ and $K = L_K L_K^*$), and δM, δC and δK. The same can be said for the new $\sin\Theta$ type theorems, for which one needs additional information on relative gaps.

More to the point, we consider the following generalized eigenvalue/eigenvector problem

$$Ax = \lambda Bx,$$

and the corresponding perturbed problem

$$\widetilde{A}\widetilde{x} = \widetilde{\lambda}\widetilde{B}\widetilde{x},$$

where $(\widetilde{A}, \widetilde{B})$ is also a definite Hermitian matrix pair and the matrices B and $\widetilde{B}$ have the same inertia.

The corresponding matrix pairs (A, B) and $(\widetilde{A}, \widetilde{B})$ can be simultaneously diagonalized (see for example [26] or [20]), that is, there exist nonsingular matrices X and $\widetilde{X}$ such that

$$X^*AX = \Lambda_A \quad \text{and} \quad X^*BX = J, \qquad \widetilde{X}^*\widetilde{A}\widetilde{X} = \widetilde{\Lambda}_A \quad \text{and} \quad \widetilde{X}^*\widetilde{B}\widetilde{X} = J, \tag{1}$$

where $\Lambda_A = \operatorname{diag}(l_1, \ldots, l_n)$, $\widetilde{\Lambda}_A = \operatorname{diag}(\widetilde{l}_1, \ldots, \widetilde{l}_n)$ and $J = \operatorname{diag}(\pm 1, \ldots, \pm 1)$. The perturbation is small enough so that the inertia of B and $\widetilde{B}$ is the same. Note that the eigenvalues of the matrix pair (A, B) are diagonal entries of the matrix $\Lambda_A J$, which will be denoted by

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \quad \text{where} \quad \lambda_i = \pm l_i, \quad i = 1, \ldots, n, \tag{2}$$

and similarly, for perturbed quantities, the eigenvalues of the matrix pair $(\widetilde{A}, \widetilde{B})$ are diagonal entries of the matrix $\widetilde{\Lambda}_A J$, which will be denoted by

$$\widetilde{\Lambda} = \operatorname{diag}(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_n), \quad \text{where} \quad \widetilde{\lambda}_i = \pm \widetilde{l}_i, \quad i = 1, \ldots, n. \tag{3}$$

The topic under consideration has been intensively studied in the last twenty years. Thus we will point out just several papers which have treated a similar issue and we find them interesting from our point of views.

Relative perturbation theory for the Hermitian pairs (A, B), where both A and B are positive definite, was given by Demmel and Veselić in [7]. Similar results were presented in [28,18] for the Hermitian pairs (A, B), where B is positive definite. These results were later generalized by Nakić in [19] on the class of so-called strongly definitizable matrix pairs. All these results are related to structured (pointwise) perturbations. We did not find an easy way to apply them to the hyperbolic eigenvalue problem (HEP). For that purpose, we generalize and improve some results from [9,10].

A nice overview of results from the relative perturbation theory point of view can be found in [14], while in [4] the similar issue was considered using standard (absolute) perturbation theory. Furthermore, in [2], Barlow and Slapničar obtained sharp relative perturbation bounds for the eigenvalues as well as the eigenspaces for the Hermitian pairs (A, B), where B can be semi-definite. In [13], the authors showed the upper bounds for the variation of generalized eigenvalues of definite Hermitian matrix pairs under perturbation.

The main drawback of the last two approaches is that for the proper application of these bounds one needs a complete eigendecomposition of an unperturbed pair (A, B).

At the same time, there exist many results connected with polynomial eigenvalue problems, especially with quadratic and hyperbolic problems. A nice survey on the quadratic eigenvalue problem, treating its many applications, can be found in [23]. Further, in [27], Veselić presented a sharp bound for the perturbations of eigenvalues of the HEP, but for the application of this bound, one needs again all eigenvectors of the triple (M, C, K). The same property holds for spectral perturbation theory of matrix polynomials which can be found in [22,6].

To the authors' knowledge, there was no straightforward way to generalize some of the above mentioned results on relative perturbation bounds for eigenvalue and eigensubspace perturbation for the HEP. Thus, the rest of the paper is devoted to the derivation of bounds which will allow us to fulfill this task.

The paper is organized as follows. The first part of this paper contains relative perturbation bounds for the eigenvalues and eigenvectors of the definite matrix pairs (A, B) and $(\widetilde{A}, \widetilde{B})$. In Section 2.1, we present a new upper bound for the relative error in eigenvalues, that is, we present an upper bound for

$$\frac{|\widetilde{\lambda}_i - \lambda_i|}{|\lambda_i|}, \qquad i = 1, \ldots, n.$$

As the second result, in Section 2, we present a new relative perturbation bound for the sines of canonical angles between unperturbed and perturbed unitary basis for the unperturbed and perturbed eigensubspaces spanned by the columns of matrices X and $\widetilde{X}$ defined similarly to (1). Since our aim is to apply these bounds to the HEP, we will derive additional bounds for the case when $B = J$ and $\widetilde{B} = J + \delta J$, where $J = \operatorname{diag}(\pm 1, \ldots, \pm 1)$.

In the second part of the paper (Section 3), we apply the above mentioned results to the hyperbolic eigenvalue problem of the form

$$(\mu^2 M + \mu C + K)v = 0,$$