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Elastic collisions among peakon solutions for the Camassa–Holm equation

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A B S T R A C T

The purpose of this paper is to study the dynamics of the interaction among a special class of solutions of the one-dimensional Camassa–Holm equation. The equation yields soliton solutions whose identity is preserved through nonlinear interactions. These solutions are characterized by a discontinuity at the peak in the wave shape and are thus called peakon solutions. We apply a particle method to the Camassa–Holm equation and show that the nonlinear interaction among the peakon solutions resembles an elastic collision, i.e., the total energy and momentum of the system before the peakon interaction is equal to the total energy and momentum of the system after the collision. From this result, we provide several numerical illustrations which support the analytical study, as well as showcase the merits of using a particle method to simulate solutions to the Camassa–Holm equation under a wide class of initial data.

1. Introduction

The purpose of this paper is to investigate the dynamics of the interaction among peakon solutions for the one-dimensional (1-D) Camassa–Holm (CH) equation as well as showcase the merits of using particle methods to simulate solutions to the CH equation using arbitrary smooth initial data. To this extent, the CH equation is given by

$$m_t + um_x + 2mu_x = 0, \quad m = u - \alpha^2 u_{xx}, \tag{1}$$

which is subjected to the following initial data:

$$m(x,0) = m_0(x). \tag{2}$$

Here, m is the momentum related to the fluid velocity u by the 1-D Helmholtz operator (see (1)).

Eq. (1) arises in a wide range of scientific applications and, for example, can be described as a bi-Hamiltonian model in the context of shallow water waves, see [2,14,15]. It can also be used to quantify growth and other changes in shape, such as those which occur in a beating heart, by providing the transformative mathematical path between two shapes (for instance, see [21] p. 420).

The CH equation exhibits some interesting properties among a class of nonlinear evolutionary PDEs. For instance, the equation is completely integrable and thus possesses an infinite number of conservation laws. Eq. (1) yields soliton solutions—whose identity is preserved through nonlinear interactions—which are characterized by a discontinuity at the peak in the wave shape., see, e.g. [2,4,27]. More precisely, Eq. (1) admits traveling wave solutions of the form

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$u(x,t) = ae^{-|x-ct|}$ with speed proportional to amplitude. For these reasons, soliton solutions generated from the CH equation are referred to as peakons. Peakons are also orbitally stable as their shape is maintained under small perturbations; see, e.g. [16,25,17].

Simulating these peakon solutions numerically poses quite a challenge—especially if one is interested in considering a peakon–antipeakon interaction (i.e., the interaction between positive and negative peakons). Several sophisticated numerical methods in finite-difference, finite-element, and spectral settings have been proposed for accurately resolving the CH equation—in particular, peakon interactions. For example, in [18], a self-adaptive mesh method was proposed, whereas in [22,23], a spectral projection method was used to simulate solutions to the CH equation. In [13], the authors used multi-symplectic integration, and in [26], an energy-conserving Galerkin scheme was proposed. In [12], the authors considered a dispersion-relation-preserving algorithm. For additional numerical schemes proposed for solving the CH equation, we refer the reader to [1,3,20,29,30] and references therein. Many of these methods are computationally intensive and require very fine grids along with adaptivity techniques in order to model the peakon behavior. Moreover, many of these methods are unable to successfully resolve the peakon–antipeakon interaction.

Solutions of (1), (2) can be accurately captured by using a particle method, as shown in [20,10,5,6]. In the particle method, described in [10,6,11], the solution is sought as a linear combination of Dirac distributions, whose positions and coefficients represent locations and weights of the particles, respectively. The solution is then found by following the time evolution of the locations and the weights of these particles according to a system of ODEs obtained by considering a weak formulation of the problem. The particle methods presented in [5,6] have been derived using a discretization of a variational principle and provide the equivalent representation of the ODE particle system. The main advantage of particle methods is their (extremely) low numerical diffusion that allows one to capture a variety of nonlinear waves with high resolution; see, e.g., [7–9,28] and references therein.

In this paper, we apply the particle method for the numerical solution of the CH equation in order to study the elastic collisions among peakon solutions. We begin, in Section 2, with a brief overview of the particle method and some of its main properties which are necessary for the study of numerical collisions among peakon solutions. We then provide in Section 3 an analytical discussion about the behavior of peakon interactions for two positive peakons. Finally, in Section 4, we present several numerical experiments which showcase both the complex interactions among peakon solutions, as well as the merits of using a particle method to simulate such solutions.

2. Description of the particle method for the Camassa–Holm equation

In this section, we briefly describe a particle method for the CH equation. For a more detailed description on the particle method for (1), we refer the reader to [10,11,6]. We begin by searching for a weak solution in the form of a linear combination of Dirac delta distributions. In particular, we look for a solution of the form:

$$m^N(x,t) = \sum_{i=1}^{N} p_i(t)\delta\big(x - x_i(t)\big). \tag{3}$$

Here, $x_i(t)$ and $p_i(t)$ represent the location of the i-th particle and its weight, and N denotes the total number of particles. The solution is then found by following the time evolution of the locations and the weights of the particles according to the following system of ODEs:

$$\begin{cases} \dfrac{dx_i(t)}{dt} = u\big(x_i(t),t\big), & i = 1,\dots,N, \\ \dfrac{dp_i(t)}{dt} + u_x\big(x_i(t),t\big)p_i(t) = 0, & i = 1,\dots,N. \end{cases} \tag{4}$$

Using the special relationship between m and u given in (1), one can directly compute the velocity u and its derivative, by the convolution $u = G * m$, where G is the Green's function

$$G\big(|x-y|\big) = \frac{1}{2\alpha}e^{-|x-y|/\alpha}, \tag{5}$$

associated with the one-dimensional Helmholtz operator in (1). Thus we have the following exact expressions for both $u(x,t)$ and (by direct computation) $u_x(x,t)$:

$$u^N(x,t) = \big(m^N * G\big)(x,t) = \frac{1}{2\alpha}\sum_{i=1}^{N} p_i(t)e^{-|x-x_i(t)|/\alpha}, \tag{6}$$

$$u_x^N(x,t) = \big(m^N * G_x\big)(x,t) = -\frac{1}{2\alpha^2}\sum_{i=1}^{N} p_i(t)\,\mathrm{sgn}\big(x - x_i(t)\big)e^{-|x-x_i(t)|/\alpha}. \tag{7}$$

To initialize the particle method for the CH equation, one should choose the initial positions of particles, $x_i(0)$, and the weights, $p_i(0)$, so that (3) represents a high-order approximation to the initial data $m_0(x)$ in (1). The latter can be done in