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Wave propagation in advected acoustics within a non-uniform medium under the effect of gravity

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ABSTRACT

We investigate linear wave propagation in non-uniform medium under the influence of gravity. Unlike the case of constant properties medium here the linearized Euler equations do not admit a plane-wave solution. Instead, we find a “pseudo-plane-wave”. Also, there is no dispersion relation in the usual sense. We derive explicit analytic solutions (both for acoustic and vorticity waves) which, in turn, provide some insights into wave propagation in the non-uniform case.

1. Introduction

The understanding of wave propagation is of great interest in several areas of science and technology, such as geophysics, electromagnetism, and in particular advected acoustics. Rarely can one obtain analytic solutions to such problems. Good computational approaches try to be guided as much as possible by analytic knowledge of the nature of the phenomenon. Thus, for example, numerical solutions of the Navier–Stokes equations rely on knowledge gained from boundary layer theory.

This paper is concerned with wave propagation in moving media. This is referred to in the literature as advected acoustics. These waves are described by linear perturbations to the main flow. Most papers are concerned with “pure” acoustic waves, by which they mean perturbations to the mean density (pressure). These pressure perturbations are described by a wave equation (with low order terms in the case of advected acoustics) for the density. Those concerned with pure acoustic phenomena do not usually examine the perturbations to the velocity field which can be recovered from the linearized Euler equations by using the pressure field already found. This approach has existed since the 19th century; see Lamb [13], Landau and Lifshitz [14] and others.

An alternative approach is to consider the Euler equations of motion, find a main flow that satisfies them, and linearize the non-linear Euler system around that main flow. We then have a set of linear partial differential equations for the density and velocity fields. Based on thermodynamic considerations one may assume that the fluid is polytropic, ($p = p(\rho)$) and this allows the closure of the equation set.

Although this paper considers wave propagation in non-uniform media, it will be useful to review at this point the known results in the case of advected acoustics in a uniform medium (constant density, ρ_0) with a main two dimensional subsonic flow moving at a constant horizontal speed in the $\mathbf{x}$-direction, u_0.

The dimensionless linearized equations take the form:

$$\frac{\partial \rho}{\partial t} + M\frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

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$$\frac{\partial u}{\partial t}+M\frac{\partial u}{\partial x}+\frac{\partial \rho}{\partial x}=0$$
$$\frac{\partial v}{\partial t}+M\frac{\partial v}{\partial x}+\frac{\partial \rho}{\partial y}=0 \tag{1}$$

where, ρ, u and v are the dimensionless perturbed density, **x**-velocity and **y**-velocity respectively. $M<1$ is the Mach number of the main flow. The above set of partial differential equations admits a plane wave solution of the form

$$\begin{pmatrix}\rho\\u\\v\end{pmatrix}=\begin{pmatrix}q_1\\q_2\\q_3\end{pmatrix}e^{i\omega(t-\alpha x-\beta y)}, \tag{2}$$

where the q_j's ($j=1,2,3$) are constants and $\omega\alpha$ and $\omega\beta$ are the dimensionless wave numbers. Substitution of this ansatz in the p.d.e.'s leads to a homogeneous set of three algebraic equations for q_1, q_2, and q_3. The solvability condition for this algebraic set is the vanishing of its determinant, which leads to the following relation:

$$(1-\alpha M)\left[(1-\alpha M)^2-\alpha^2-\beta^2\right]=0. \tag{3}$$

The second factor, namely

$$(1-\alpha M)^2=\alpha^2+\beta^2 \tag{4}$$

is the usual dispersion relation for acoustic waves in a main flow moving at Mach number M and its plane wave solution is

$$\begin{pmatrix}\rho\\u\\v\end{pmatrix}=\begin{pmatrix}1-M\alpha\\\alpha\\\beta\end{pmatrix}e^{i\omega(t-\alpha x-\beta y)}. \tag{5}$$

The first factor in the solvability condition, namely

$$(1-\alpha M)=0, \tag{6}$$

leads to vorticity waves of the form

$$\begin{pmatrix}\rho\\u\\v\end{pmatrix}=\begin{pmatrix}0\\-\beta M\\1\end{pmatrix}e^{i\omega(t-\frac{1}{M}x-\beta y)}. \tag{7}$$

Vorticity waves behave differently from acoustic waves: their speed is that of the main flow rather than being related to the speed of sound. Also, they are associated with zero density and pressure perturbations. These results are delineated in a paper (to be submitted) which includes experimental verification of the relevant theoretical predictions. The experiments were carried out in the quiet wind tunnel at the School of Engineering, Tel Aviv University. The theoretical results concerning vorticity appeared previously, see [1,6,2].

In the present paper we consider the case of non-uniform medium, but still with a constant main flow. Specifically, we investigate the propagation of waves in the terrestrial atmosphere where the main density profile is caused by gravity.

In Section 2 we derive the appropriate equations for this case. We show that the resulting set of linear partial differential equations does not admit the usual plane-wave solution, (2), i.e. with a constant coefficient vector $\mathbf{q}=(q_1,q_2,q_3)^T$. Instead, we propose a generalization of ansatz (2) which we call a pseudo plane wave, with a coefficient vector $\mathbf{q}=\mathbf{q}(y)$. This ansatz leads to a set of ordinary differential equations, rather than an algebraic set. We present a closed form solution for both the acoustic and vorticity waves. We then deduce from these solutions certain properties of the wave propagation phenomena under consideration herein.

In Section 3 we present a numerical example that validates a prediction implied by the pseudo-plane-wave model.

In the summary we discuss insights gained and future work.

2. Analysis

In this section we present the analysis of the case of advected wave propagation in the terrestrial atmosphere.

We start with the full 2-D Euler equations under the influence of gravity:

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0$$
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y}=-g \tag{8}$$