Applied Numerical Mathematics 93 (2015) 87–106

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

High-order accurate difference potentials methods for parabolic problems

Jason Albright, Yekaterina Epshteyn *, Kyle R. Steffen

Department of Mathematics, The University of Utah, Salt Lake City, UT, 84112, United States

A R T I C L E   I N F O

Article history:
Available online 22 August 2014

Keywords:
Difference potentials
Boundary projections
Cauchy's type integral
Parabolic problems
Variable coefficients
Heterogeneous media
High-order finite difference schemes
Difference Potentials Method
Immersed Interface Method
Interface/composite domain problems
Non-matching interface conditions
Non-matching grids
Parallel algorithms

A B S T R A C T

Highly-accurate numerical methods that can efficiently handle problems with interfaces and/or problems in domains with complex geometry are crucial for the resolution of different temporal and spatial scales in many problems from physics and biology. In this paper we continue the work started in [8], and we use modest one-dimensional parabolic problems as the initial step towards the development of high-order accurate methods based on the Difference Potentials approach. The designed methods are well-suited for variable coefficient parabolic models in heterogeneous media and/or models with non-matching interfaces and with non-matching grids. Numerical experiments are provided to illustrate high-order accuracy and efficiency of the developed schemes. While the method and analysis are simpler in the one-dimensional settings, they illustrate and test several important ideas and capabilities of the developed approach.

1. Introduction

Designing numerical methods with high-order accuracy for problems with interfaces (for example, models for composite materials or fluids, etc.), as well as models in domains with complex geometry is crucial to many physical and biological applications. Moreover, interface problems result in non-smooth solutions (or even discontinuous solutions) at the interfaces, and therefore standard numerical methods (finite-difference, finite-element methods, etc.) in any dimension (including 1D) will very often fail to produce accurate approximation of the solutions to the interface problems, and thus special numerical algorithms have to be developed for the approximation of such problems (for instance, see simplified 1D example of interface problem in [8], page 12 and Table 7 on page 14).

There is extensive literature that addresses problems in domains with irregular geometries and interface problems. Among finite-difference based methods for such problems are the Immersed Boundary Method (IB) ([24,25], etc.), the Immersed Interface Method (IIM) ([14–16,1,12], etc.), the Ghost Fluid Method (GFM) ([9,17,18,10], etc.), the Matched Interface and Boundary Method (MIB) ([42,39,41,40], etc.), and the method based on the Integral Equations approach ([20], etc.). Among the finite-element methods for interface problems are ([2,4,35,22,38,37], etc.). These methods are robust sharp interface methods that have been applied to solve many problems in science and engineering. For a detailed review of the subject the reader can consult, for example, [16]. *However, in spite of great advances in the numerical methods (finite-difference, finite-element, etc.) for interface problems it is still a challenge to design high-order accurate methods for such problems. To the best of our knowledge, there are currently only a few high order (higher than second order in space) schemes for parabolic interface problems*

* Corresponding author.

http://dx.doi.org/10.1016/j.apnum.2014.08.002
0168-9274/

[10]. In [10], the high-order (fourth-order in space) GFM is constructed for the 2D heat equation under the assumption of the Dirichlet boundary conditions at the interfaces, and extended (as the third-order method) to the Stefan problem as well. Note, that the method in [10] is developed for the piecewise constant coefficients problems.

We develop here an approach based on the Difference Potentials Method (DPM) [28,31] (see about DPM for example in, [28,31,11,19,32,36,21,23,29,34,6–8], etc.). The DPM allows one to reduce uniquely solvable and well-posed boundary value problems to pseudo-differential boundary equations with projections. Methods based on Difference Potentials ([28, 29,34,7,8,21,30], etc.) introduce computationally simple auxiliary domains. After that, the original domains/subdomains are embedded into simple auxiliary domains (and the auxiliary domains are discretized using Cartesian grids). Next, methods based on Difference Potentials construct discrete pseudo-differential *Boundary Equations with Projections* to obtain the values of the solutions at the points near the continuous boundaries of the original domains (at the points of the discrete grid boundaries which approximate the continuous boundaries from the inside and outside of the domains). Using the obtained values of the solutions at the discrete grid boundaries, the approximation to the solution in each domain/subdomain is constructed through the discrete generalized Green's formulas.

The main complexity of methods based on Difference Potentials reduces to several solutions of simple auxiliary problems on structured Cartesian grids. Methods based on Difference Potentials approach are not restricted by the type of the boundary or interface conditions (as long as the continuous problems are well-posed), and are also computationally efficient since any change of the boundary/interface conditions affects only a particular component of the overall algorithm, and does not affect most of the numerical algorithm (see [28], or some example of the recent works [3,29,34,7,8], etc.). Finally, unlike many existing finite-difference based methods for interface problems, the Difference Potentials approach is well-suited for the development of parallel algorithms for such problems, see [29,34,7] – examples of the second-order in space schemes based on the Difference Potentials for 2D interface/composite domain problems and see Section 4 below. The reader can consult [28,31] and [26,27] for a detailed theoretical study of the methods based on Difference Potentials, and ([28,31,19, 32,36,33,21,3,13,11,30,23,29,34,6–8], etc.) for the recent developments and applications of DPM.

In this paper, we extend the work on high-order methods started in [8] *to variable coefficient parabolic models.* We begin here with the modest consideration of one-dimensional variable coefficient parabolic interface models, and we develop and numerically test high-order accurate methods based on Difference Potentials methodology. *At this point we are not aware of any other high-order (higher than second-order in space) method for the parabolic interface problems in heterogeneous media. Moreover, numerical experiments in Section 6 indicate that the developed method preserves high-order accuracy on the interface problems (including problems with discontinuous diffusion coefficients at the interface and jump conditions in the solution at the interface), not only in the solution, but also in the discrete gradient of the solution. To the best of our knowledge, the present work is also the first extension (at this point, in modest 1D settings) of the Difference Potentials approach for the construction of high-order accurate numerical schemes for parabolic problems.* Although, the method and analysis are simpler in the current one-dimensional settings, they illustrate and test several important ideas and abilities of the Difference Potentials approach with application to interface problems. Let us note that, previously in [7], we have developed an efficient (second-order accurate in space and first-order accurate in time) scheme based on Difference Potentials approach for 2D interface/composite domain constant coefficient parabolic problems. The second-order in space method developed in [7] can handle non-matching interface conditions on the solution (as well as non-matching grids between each subdomain), and is well-suited for the design of parallel algorithms. However, it was constructed and tested for the solution of the heat equation in irregular domains and/or with interfaces.

The paper is organized as follows. In Section 2, we introduce the formulation of the problem. Next, to illustrate the unified approach behind the construction of DPM with different orders of accuracy, we construct DPM with second and with fourth-order accuracy in space in Section 3.1 for a single domain 1D parabolic model. In Section 4, we extend the developed methods to one-dimensional parabolic interface/composite domain model problems. In Section 5 for the reader's convenience we give a brief summary of the main steps of the presented algorithms. Finally, we illustrate the performance of the proposed Difference Potentials Methods, as well as compare Difference Potentials Methods with the Immersed Interface Method, in several numerical experiments in Section 6. Some concluding remarks are given in Section 7.

2. Parabolic interface models

We are concerned in this work with a 1D parabolic interface (with fixed interface at this point) problem of the form: denote, $u_t - L_1[u] \equiv u_t - (k_1 u_x)_x$, and $u_t - L_2[u] \equiv u_t - (k_2 u_x)_x$, thus

$$u_t - L_1[u] = f_1, \quad x \in I_1, \tag{2.1}$$

$$u_t - L_2[u] = f_2, \quad x \in I_2, \tag{2.2}$$

subject to the Dirichlet boundary conditions specified at the points $x = 0$ and $x = 1$:

$$u(0,t) = a(t), \quad \text{and} \quad u(1,t) = b(t), \tag{2.3}$$

interface conditions at α:

$$\beta_1 u_1(\alpha,t) - \beta_2 u_2(\alpha,t) = \phi(t), \qquad \tilde{\beta}_1 u_{1x}(\alpha,t) - \tilde{\beta}_2 u_{2x}(\alpha,t) = \tilde{\phi}(t) \tag{2.4}$$