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Stability criteria for non-self-adjoint finite differences schemes in the subspace

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Dedicated to Professor V.S. Rjaben'kii on the occasion of his 90th birthday

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ABSTRACT

The finite differences schemes with weights for the heat conduction equation with nonlocal boundary conditions $u(0,t)=0$, $\gamma \frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(1,t)$ are discussed, where γ is a given real parameter. On some interval $\gamma \in (\gamma_1,\gamma_2)$ the spectrum of the differential operator contains three eigenvalues in the left complex half-plane, while the remaining eigenvalues are located in the right half-plane. Earlier only the case of one eigenvalue λ_0 located in the left half-plane was considered. The stability criteria of finite differences schemes is formulated in the subspace induced by stable harmonics.

1. Introduction

The famous V.S. Rjaben'kii and A.F. Filippov papers [6,16,17] marked the beginning of the new stage in the stability theory of differences schemes. Specifically, it largely stimulated the development of the theory of two-layer and three-layer operator-differences schemes in Hilbert space [18,19]. The generality of the notion of differences schemes stability was stressed in [6,16,17], as well as the opportunity to study a large variety of differences schemes from a unified point of view. Besides that, it was shown that for general linear problems the convergence of the finite differences scheme follows from approximation and stability. Thus, the study of stability is one of the main problems in the finite differences schemes theory (see also [12]).

The certain difficulties arise in the investigation of stability of non-self-adjoint operator-differences schemes. In this case not only general theorems are of great interest, but also the formulation of stability criteria for specific differences schemes is also important. The present paper deal with detailed consideration with one of such examples.

Mathematical physics problems with nonlocal boundary conditions and their finite differences approximations attract the attention of many researchers, starting from the classical papers of A.V. Bitsadze and A.A. Samarskii [4]. N.I. Ionkin [14,15] began to study the differences schemes for such problems, based on the results of V.A. Il'in [13]. A detailed overview of the papers up to 2008 is given in [8,11], dedicated to the differences methods for problems with nonlocal boundary conditions. Let us note papers [3,5,22–25], where the problems with boundary conditions of integral type and multi-dimensional problems are studied. In the papers [1,2] a method of energy inequalities was developed to obtain a priori estimations of the solution of finite differences schemes for equations with variable coefficients.

Let us consider the heat conduction equation with nonlocal boundary conditions

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, \quad 0<x<1, \quad u(x,0)=u_0(x),$$

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$$u(0,t)=0, \qquad \gamma \frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(1,t), \tag{1}$$

where γ is a given real parameter. Physical interpretation of such tasks, as well as their purely mathematical research can be found for example in the monograph of V. Steklov [26]. While solving (1) by method of separation of variables, we arrive at the eigenvalue problem

$$X''(x)+\lambda X(x)=0, \quad 0<x<1, \qquad X(0)=0, \qquad \gamma X'(0)=X'(1). \tag{2}$$

The spectrum of problem (2) depends on the parameter γ:

$$\lambda_0=\psi^2, \quad \psi=\arccos\gamma,$$
$$\lambda_{2k-1}=(2\pi k-\psi)^2, \qquad \lambda_{2k}=(2\pi k+\psi)^2, \quad k=1,2,\ldots$$

If $|\gamma|\leqslant 1$ all eigenvalues are real and positive. If $\gamma>1$ the eigenvalues are complex, and $\psi=i\ln(\gamma+\sqrt{\gamma^2-1})$. In addition, when $\gamma>1$ there are eigenvalues with negative real part.

The number of eigenvalues located in the left half-plane increases with the growth of γ. It is known that if $1<\gamma<\gamma_1=\cosh(2\pi)$ there exists a unique eigenvalue $\lambda_0=\psi^2$, located in the left half-plane. If $\gamma_1<\gamma<\gamma_2=\cosh(4\pi)$ there is real eigenvalue $\lambda_0<0$ and there are two complex conjugate eigenvalues λ_1 and λ_2 in the left half-plane.

In the case $\gamma\in(1,\gamma_2)$ the problem (1) is unstable with respect to the initial data, since the harmonics related to the eigenvalues λ_0, λ_1 and λ_2, infinitely grow up in time. But the remaining harmonics $X^{(k)}(x)\exp(-\lambda_k t)$, $k=3,4,\ldots$ are limited and even tend to zero as $t\to\infty$. In this sense, the problem (1) can be called stable in the subspace of functions of the form $\sum_{k=3}^{\infty} c_k X^{(k)}(x)$.

Everywhere below we use the formulation of basic concepts and notations of the theory of finite differences schemes of A.A. Samarskii [18].

Let us approximate the problem (1) by the finite differences scheme with weights:

$$y^n_{t,i}-y^{(\sigma)}_{\bar{x}x,i}=0, \quad i=1,2,\ldots,N-1, \ n=0,1,\ldots,$$
$$y^0_i=u_0(x_i), \qquad y^{n+1}_0=0, \qquad \frac{h}{2}y^n_{t,N}+y^{(\sigma)}_{\bar{x},N}-\gamma y^{(\sigma)}_{x,0}=0. \tag{3}$$

In this formula $y^n_i=y(x_i,t_n)$, $x_i=ih$, $t_n=n\tau$ and

$$y^{(\sigma)}_i=\sigma y^{n+1}_i+(1-\sigma)y^n_i, \qquad y^n_{t,i}=\frac{y^{n+1}_i-y^n_i}{\tau}, \qquad y^n_{\bar{x},i}=\frac{y^n_i-y^n_{i-1}}{h}, \qquad y^n_{x,i}=\frac{y^n_{i+1}-y^n_i}{h}.$$

Note that the differential boundary condition is of the order of approximation to the solution of the original problem $O(\tau+h^2)$ $\sigma\neq 0.5$ $O(\tau^2+h^2)$ $\sigma=0.5$. The same holds true for the main difference equations.

Finite differences schemes for the problem (1) were considered originally by N.I. Ionkin [14,15] (see also [11, pp. 10–65]) for the case $\gamma=1$ (so-called problem of Samarskii–Ionkin). The case of $\gamma\in(-1,1)$ has been studied in [11, p. 158]. A fundamental differences from the case $\gamma=\pm 1$ is that the system of eigenfunctions of the main differences operator of problem (3) with $\gamma\neq\pm 1$ constitutes a basis in the space of grid functions.

Finite differences schemes corresponding to $\gamma\in(1,\gamma_1)$ are considered in [10]. In the present paper we study the behavior of the finite differences schemes for $\gamma\in(\gamma_1,\gamma_2)$, where $\gamma_1=\cosh(2\pi)$, $\gamma_2=\cosh(4\pi)$.

Let us comment the notations we use in this paper. We use the parameters: $\tau>0$ is a time-step, N is the number of points with respect to the spatial variable, $h=1/N$ and $\gamma>1$. For the clarity, *we suppose the number N is even*. Clearly, this restriction can be eliminated with a little complication of symbols. Next we denoted $\alpha=\ln(\gamma+\sqrt{\gamma^2-1})$, $a=\cosh(h\alpha)$, $\kappa=\tau/h^2$, σ is the weight factor. Additionally k is the number of the eigenvalue, $z_k=\cos(2\pi kh)$, $a_k=z_k^{-1}$, $r_k(a)=1-az_k$, $s_k(a)$ is an eigenvalue of transition operator.

The solution of the differences problem (3) is an element of a complex linear N-dimensional space H, consisting of vectors $y=(y_1y_2\cdots y_N)^{\mathrm{T}}$, where $y_i=y(x_i)$. The main operator $A:H\to H$ of the differences scheme (3) is the operator of the second finite differences derivative with the nonlocal boundary conditions. Eigenvalues of the operator A are numbered as λ_{2k-1} $(k=1,2,\cdots,N/2)$ and λ_{2k} $(k=0,1,\cdots,N/2-1)$. The total enumeration $\{\lambda_l\}_{l=0}^{N-1}$ also is used.

All eigenvalues are simple, $\lambda_0<0$, $\lambda_{N-1}>0$, and λ_{2k-1} and λ_{2k} are complex conjugate numbers with a non-zero imaginary part. Let $\mu^{(l)}$ be eigenvector of the operator A, corresponding to the eigenvalue λ_l. The system of eigenvectors $\{\mu^{(l)}\}_{l=0}^{N-1}$ constitutes a basis in the space H.

Next we consider the square matrix $M=[\mu^{(0)}\mu^{(1)}\cdots\mu^{(N-1)}]$, the columns of this matrix are the coordinates of the vectors $\mu^{(l)}$ in the unit basis. We consider also the subspaces $H^{(k)}(a)\subset H$ that are defined as follows. Let $H^{(0)}(a)$ and $H^{(N/2)}(a)$ be one-dimensional subspaces of vectors collinear to $\mu^{(0)}$ and $\mu^{(N/2)}$ respectively. Let $H^{(k)}(a)$ $(k=1,2,\ldots,N/2-1)$ be two-dimensional subspaces spanned on the complex conjugate eigenvectors $\mu^{(2k-1)}$ and $\mu^{(2k)}$.

All eigenvalues and their eigenvectors depend on the parameter $a=\cosh(h\ln(\gamma+\sqrt{\gamma^2-1}))$. In the present paper, we consider the case when $\operatorname{Re}\lambda_0(a)<0$, $\operatorname{Re}\lambda_1(a)=\operatorname{Re}\lambda_2(a)<0$ and $\operatorname{Re}\lambda_l(a)\geqslant 0$ for $l=3,4,\cdots,N-1$. The investigation of stability of the finite differences scheme is performed in the subspace $H_3^{N-1}(a)$ spanned on the vectors $\{\mu^{(l)}(a)\}_{l=3}^{N-1}$ related to eigenvalues with a non-negative real part.