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Uzawa-like methods for numerical modeling of unsteady viscoplastic Bingham medium flows

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ABSTRACT

The Uzawa-like algorithm is implemented for two-dimensional flows of viscoplastic fluids. The rheological model employed is the ideal Bingham model. As a test the lid-driven square-cavity benchmark problem is considered. The results for the steady-state problem are faithfully reproduced as compared to those in the literature for the shape and location of the yield surface. The proposed method is very successful at capturing both yielded and unyielded regions.

1. Introduction

Viscoplastic materials behave as rigid solids when the imposed stress is smaller than the yield stress, and they flow as fluids when yielded. The flow field is divided into two regions: the unyielded (rigid) and the yielded (fluid) zone. As a rule, the following two types of rigid zones are traditionally distinguished: the stagnation (dead) zones, where the medium is at rest, and the plug region (core), where the medium moves as a rigid body. The separation surfaces between rigid and fluid zones are related to the yield surfaces. The location and shape of the unyielded region have to be found as part of the solution of the initial boundary-value problem. Thus, the characteristic feature in the problem of viscoplastic medium flows is the necessity of constructing a solution in a domain with an unknown boundary.

The main difficulty in the numerical simulation of viscoplastic fluid flow is related to the non-differentiable form of constitutive law and inability to evaluate the stresses in regions where the material has not yielded. There are two principal approaches that have been proposed in the literature to overcome the mathematical problem of viscoplastic fluid flow. The first one, known as regularization method, consists in approximating the constitutive equation by a smoother one. The second method is based on the theory of variational inequalities [4]. In the latter case, the problem reduces to the minimization of a functional and a further solution of the equivalent saddle-point problem. To solve the saddle-point problem have been proposed two slightly different methods. The Uzawa-like method [9], based on the multiplier theorem [4], introduces an additional variable (multiplier), which is proportional to the tensor of plastic stress. Augmented Lagrangian method (ALM) [6] introduces additional strain rate tensor and Lagrange multiplier, which has the meaning of the extra stress tensor. Various contributions to the literature include [2,6,8,10,16,17,21–23]. For the numerical simulation of Bingham flow taking into account the convection time discretization by operator-splitting (fractional step) [13] is used. The projection method is a form of the fractional step method adapted to the Navier–Stokes equations. A large body of literature has been devoted to the construction, analysis and implementation of projection-type schemes [12]. The ancestor is the Chorin–Temam [18] projection method which is based on the Helmholtz decomposition principle. Fractional step methods have employed to decouple three main difficulties of the simulation of the Bingham fluid flow: the nonlinear convective term, the non-differentiability of the constitutive law, and the incompressibility assumption.

Dean, Glowinski and Guidoboni [2] have applied the first-order operator splitting scheme for the simulation of the Bingham media flow. For the time scheme, they decoupled the main system into two sub-systems: a Navier–Stokes problem and a plasticity problem. They further decouple the Navier–Stokes problem into the generalized Stokes problem and transport

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problem. Another numerical strategy to calculate unsteady flow of Bingham fluids that relies on decoupling scheme was proposed in [22]. It consists of three steps – convective step, diffusion step and plasticity step, the latter was solved by algorithm ALG2. Third variant of operator-splitting method was presented in [23] (it is modification of method proposed in [2]). For the space discretization the finite element method [2,22] and the finite difference method on half-staggered grids [23] were used. In all paper listed above, the lid-driven cavity problem was considered as a numerical example. The main goal of current study is to suggest another operator-splitting scheme and provide computational experiments for high Reynolds numbers. For discretization we use the finite difference method [11] on staggered grids (MAC).

2. Problem statement

The isothermal flow of an incompressible viscoplastic fluid is governed by the following equations:

$$\rho\big(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\big) = -\nabla p + \nabla\cdot\boldsymbol{\tau} + \mathbf{f} \quad \text{in } \Omega\times(0,T), \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times(0,T), \tag{2}$$

$$\boldsymbol{\tau} = 2\mu\mathbf{D} + \tau_0\sqrt{2}\frac{\mathbf{D}}{|\mathbf{D}|}, \quad \text{if } |\boldsymbol{\tau}|\ge\tau_0, \tag{3}$$

$$|\mathbf{D}| = 0, \quad \text{if } |\boldsymbol{\tau}|<\tau_0.$$

Here Ω denotes a bounded domain in R^d $(d=2,3)$, Γ the boundary of the domain, $[0,T]$ a time interval, $\mathbf{u}$ is the velocity vector, ρ is the density, p is the pressure, $\boldsymbol{\tau}$ is the extra stress tensor, $\mathbf{f}$ is the given field of external forces, τ_0 is the yield stress and μ is the plastic viscosity, $\mathbf{D}$ is rate-of-strain tensor $\mathbf{D}(\mathbf{u}) = (\nabla\mathbf{u} + (\nabla\mathbf{u})^T)/2$ with norm $|\mathbf{D}| = \sqrt{\mathbf{D}:\mathbf{D}}$, and $\mathbf{A}:\mathbf{B} = \sum_{i=1}^d\sum_{j=1}^d a_{ij}b_{ij}$, for all $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$. The first equation represents the momentum equation, the second one is continuity equation, while the third one is the rheological constitutive relations (Bingham model). Hereinafter we consider $d=2$. The above system of equations must be provided with suitable initial and boundary conditions:

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega, \qquad \nabla\cdot\mathbf{u}_0 = 0, \qquad \mathbf{u} = \mathbf{u}_B \quad \text{on } \Gamma\times(0,T). \tag{4}$$

Following Duvaut and Lions [4], instead of nonlinear equations (1)–(4) we consider the following variational inequality model: find $\mathbf{u}\in(H_0^1(\Omega))^2$ such that a.e. on $(0,\ T)$ we have

$$\begin{aligned}&\rho\int_\Omega \partial_t\mathbf{u}(t)\cdot\big(\mathbf{v}-\mathbf{u}(t)\big)\mathrm{d}\mathbf{x} + \rho\int_\Omega\big(\mathbf{u}(t)\cdot\nabla\big)\mathbf{u}(t)\cdot\big(\mathbf{v}-\mathbf{u}(t)\big)\mathrm{d}\mathbf{x}\\ &\quad+\mu\int_\Omega\nabla\big(\mathbf{u}(t)\big):\nabla\big(\mathbf{v}-\mathbf{u}(t)\big)\mathrm{d}\mathbf{x} + \tau_0\sqrt{2}\int_\Omega\big(|\mathbf{D}(\mathbf{v})| - |\mathbf{D}(\mathbf{u}(t))|\big)\mathrm{d}\mathbf{x}\\ &\ge\int_\Omega\mathbf{f}(t)\cdot\big(\mathbf{v}-\mathbf{u}(t)\big)\mathrm{d}\mathbf{x}, \quad \forall\mathbf{v}\in\mathbf{U}_B,\end{aligned} \tag{5}$$

$$\nabla\cdot\mathbf{u}(t) = 0 \quad \text{in } \Omega, \qquad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega, \qquad \mathbf{u}(t) = \mathbf{u}_B(t) \quad \text{on } \Gamma,$$

$$\mathbf{U}_B = \big\{\mathbf{v}\in\big(H^1(\Omega)\big)^d \mid \mathbf{v} = \mathbf{u}_B(t) \text{ on } \Gamma\big\}. \tag{6}$$

It follows from [4] that there exists a tensor-valued function $\boldsymbol{\lambda}$ such that the formulation (5)–(6) is equivalent to

$$\boldsymbol{\tau} = 2\boldsymbol{\mu}\mathbf{D}(\mathbf{u}) + \tau_0\sqrt{2}\boldsymbol{\lambda}, \tag{7}$$

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}^T, \quad |\boldsymbol{\lambda}|\le 1, \quad \boldsymbol{\lambda}:\mathbf{D}(\mathbf{u}) = |\mathbf{D}(\mathbf{u})| \quad \text{a.e. in } \Omega\times(0,T), \tag{8}$$

$$\rho\left[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] - \mu\Delta\mathbf{u} - \tau_0\sqrt{2}\nabla\cdot\boldsymbol{\lambda} + \nabla p = \mathbf{f} \quad \text{in } \Omega\times(0,T), \tag{9}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times(0,T), \qquad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega, \qquad \mathbf{u} = 0 \quad \text{on } \Gamma\times(0,T). \tag{10}$$

Relations (7) are equivalent to $\boldsymbol{\lambda} = P_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} + r\tau_0\sqrt{2}\mathbf{D})$ for $\forall r\ge 0$ with the closed convex set Λ and the projection operator $P_\Lambda : (L^2(\Omega))^4 \to \boldsymbol{\Lambda}$,

$$\boldsymbol{\Lambda} = \big\{q\in\big(L^2(\Omega)\big)^4,\ |q|\le 1,\ q = q^T\big\},$$

$$P_\Lambda(\mathbf{q}) = \big\{\mathbf{q},\ \text{if } |\mathbf{q}|\le 1,\ \mathbf{q}/|\mathbf{q}|,\ \text{if } |\mathbf{q}|>1\big\}. \tag{11}$$

In most of the papers devoted to numerical modeling of Bingham medium, the convective terms are neglected. In this case, we use backward Euler scheme for time-discretization of problem (5). We have supposed the time interval of interest $(0,T)$ has been divided into N subintervals $[t^n, t^{n+1}]$, where $t^{n+1} - t^n = \Delta t$, $n = 0, 1, \ldots, N-1$. Assume

$$\mathbf{u}^0 = \mathbf{u}_0,$$