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High-order accurate monotone compact running scheme for multidimensional hyperbolic equations

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ABSTRACT

Monotone absolutely stable conservative difference schemes intended for solving quasilinear multidimensional hyperbolic equations are described. For sufficiently smooth solutions, the schemes are fourth-order accurate in each spatial direction and can be used in a wide range of local Courant numbers. The order of accuracy in time varies from the third for the smooth parts of the solution to the first near discontinuities. This is achieved by choosing special weighting coefficients that depend locally on the solution. The presented schemes are numerically efficient thanks to the simple two-diagonal (or block two-diagonal) structure of the matrix to be inverted. First the schemes are applied to system of nonlinear multidimensional conservation laws. The choice of optimal weighting coefficients for the schemes of variable order of accuracy in time and flux splitting is discussed in detail. The capabilities of the schemes are demonstrated by computing well-known two-dimensional Riemann problems for gasdynamic equations with a complex shock wave structure.

1. Introduction

Hyperbolic equations, specifically, hyperbolic conservation laws describe a wide variety of physical phenomena, in particular, those involving discontinuous solutions. This circumstance has motivated the development of numerous modern high-resolution numerical methods intended for finding approximate solutions of such equations. These methods can be divided into two basic types: upwind and central schemes. Most upwind schemes stem from the first-order Godunov scheme [13]. The evolution step in these schemes is based on solving a Riemann problem posed on cell boundaries. Within the framework of upwind schemes, Godunov-type algorithms have been created based on high-order reconstructions and on the TVD and TVB properties of solutions [5,13,26,27,53]. The idea of an adaptive stencil was implemented in high-order essentially nonoscillatory (ENO) reconstructions [16,17] (intended to minimize spurious oscillations near discontinuities of the solution) and in schemes based on such reconstructions. This idea was further developed in weighted essentially nonoscillatory (WENO) schemes [18,35], which make use of a convex linear combination of interpolating polynomials on several consecutive stencils. The prototype of most central schemes is the first-order Lax–Friedrichs (LxF) scheme [24]. This is an explicit two-level (in time) difference scheme with a two-point spatial stencil at the lower time level. The LxF method is conservative and monotone; therefore, this is a TVD method. Like the original Godunov method, the LxF scheme is based on a piecewise constant approximation of the solution, but it does not require solving a Riemann problem for time advancing and uses only flux estimates. However, the LxF scheme has a relatively low resolution. The simplicity of the LxF

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algorithm and the relative ease of its extension to systems of equations and multidimensional problems have motivated the development of high-resolution central schemes by applying high-order reconstructions and more accurate time integration methods. A second-order accurate central scheme was developed by Nessyahu and Tadmor [38]; it is widely known as the NT scheme. It is based on a staggered grid and uses the reconstruction of MUSCL-type piecewise linear interpolants in space, oscillation-suppressing nonlinear limiters, and the midpoint quadrature rule for evaluating integrals with respect to time. Later, the NT scheme was improved and developed in several directions. Kurganov and Tadmor [23] designed a second-order central scheme, which has a semidiscrete version with a lower numerical viscosity. Third-order nonoscillatory central schemes based on a parabolic reconstruction [20,34,36] were constructed, and central-upwind schemes [21] were designed, in which one-sided local propagation velocities were used. Bianco etc. [2] combined central schemes with third- and fourth-order accurate ENO reconstructions. Integrals with respect to time in these schemes are evaluated using a natural continuous extension of the Runge–Kutta method [64]. For one-dimensional problems, high-order CWENO central schemes were designed with the use of a polynomial WENO reconstruction in [29,30,42]. In [22,31,32] central schemes were extended to multidimensional problems. Upwind and central schemes have their own advantages and shortcomings. Generally speaking, upwind schemes ensure a better resolution near discontinuities of the solution than central schemes of the same order of accuracy with grid cell of the same size. However, upwind schemes are more expensive and more difficult to implement than central schemes, primarily because Riemann problems have to be solved on the boundaries of discontinuities in order to compute the time evolution of the solution. Moreover, the solution of these problems depends strongly on the structure of the flux function and on the form of the equation of state, which relates the physical parameters of the gas medium. This prevents the development of a universal code based on upwind schemes. In central schemes, there is no need to solve Riemann problems and they are easy to extend to systems of equations and multidimensional problems. For this reason, much attention has been given to central schemes in recent years.

When a finite-difference has a compact stencil, this provides additional advantages: the use of efficient methods for solving difference equations (tridiagonal matrix algorithm or running computations) [47,48,54,55], the convenience of setting boundary conditions [49], and a good spectral resolution [28]. A shortcoming of compact schemes is that they generate spurious oscillations (Gibbs phenomenon) near shock waves and in high-gradient regions. Various methods for eliminating or reducing such oscillations have been proposed in the literature. The basic ones are outlined below. In [4,43,56,57,61] oscillations near shock waves are suppressed by introducing special limiters of numerical fluxes into the scheme. Artificial dissipation and numerical filters are also introduced into compact schemes to reduce the oscillations near discontinuities [3,6,7,10,11,40,59,62,63]. In some works, compact schemes with an improved spectral resolution in regions of smooth solutions are combined with ENO/WENO schemes, which exhibit the nonoscillatory behavior near discontinuities of the solution. A hybrid compact ENO scheme was proposed in [1]. Hybrid compact WENO schemes can be found in [41,44,52]. An alternative to hybrid schemes is ones in which the fluxes on cell boundaries are computed using compact (Hermitian) interpolations on candidate stencils. Then an ENO algorithm is used to choose a suitable stencil or a WENO algorithm is applied to compute weighting coefficients of the compact interpolations on the candidate stencils. Such compact ENO schemes were proposed in [8], while compact WENO schemes were suggested in [9,12,19,39,65].

In [37,45,48] hybrid finite-volume–finite-difference (FV-FD) compact central schemes on nonstaggered grids were proposed for solving nonstationary one-dimensional hyperbolic equations and systems of conservation laws. On the one hand, these schemes were derived from integral conservation laws; on the other hand, they were written using grid values of the desired quantities. In FV-FD schemes, there is no need to solve Riemann problems for computing the time evolution of the solution. The schemes are fourth-order accurate for sufficiently smooth solutions in each spatial direction and can be used in a wide range of local Courant numbers. Although the compact schemes from [37,45,48] are implicit and absolutely stable, only two-diagonal matrices are to be inverted in them. A feature of these schemes is that they preserve their properties (the order of accuracy, conservativeness, and monotonicity) on an arbitrary nonuniform grid. Moreover, they preserve the monotonicity of the solution in a wide range of local Courant numbers and are easy to extend to multidimensional problems for hyperbolic conservation laws [46]. Additionally, the solution of multidimensional schemes requires the inversion of only block two-diagonal matrices. Note that the prototype of the schemes from [37,45,48] is a two-point implicit central difference scheme [60] that is second-order accurate in space.

In [46] compact schemes for two-dimensional hyperbolic conservation laws were obtained, the way of constructing the schemes for the three-dimensional equations was specified, as well as monotonicity and conservatism of the schemes were considered on examples of the solution of two-dimensional problems for the scalar linear and nonlinear transport equations. In present paper, we develop the schemes proposed in [46]. First the schemes are applied to the solution of system of multidimensional conservation laws. Two important aspects are discussed that are associated with the application of these schemes to multidimensional system of gas dynamics equations in problems with a complex shock wave structure. The schemes have a variable order of accuracy in time that varies from the third for the smooth parts of the solution to the first near discontinuities. This is achieved by introducing special weighting coefficients depending locally on the solution. We discuss how to choose optimal weighting coefficients for a scheme. The other important aspect concerning the application of the schemes from [46] is the organization of characteristic flux splitting in the numerical solution of systems of hyperbolic equations. The indicated aspects are addressed as applied to well-known one- and two-dimensional Riemann problems.