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# The mathematical modeling of the electric field in the media with anisotropic objects ☆

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ABSTRACT

We present a numerical scheme for modeling the electric field in the media with tensor conductivity. This scheme is based on vector finite element method in frequency domain. The numerical computations of the electric field in the anisotropic medium are done. The conductivity of the anisotropic medium is positive defined dense tensor in general case. We consider the electric field from anisotropic layer, inclined anisotropic layer and some anisotropic objects in isotropic half-space.



## 1. Introduction

In recent years, the vector finite element method (vector FEM) has been extensively used for modeling of electromagnetic fields [2,7,16,3,6,8,21,4]. Vector FEM allows to solve the system of Maxwell's equations at Nedelec's spaces as vector field **E** or **H**. The conditions of tangent continuity of electric field **E** and normal continuity of magnetic flux **B** are fulfilled for the FEM solution. This condition is done automatically on interior and exterior boundaries of the domains with contrast physical properties. There are the computational schemes that provide for fulfillment of conservation low in weak form [11]. These schemes guarantee computing accuracy of the electrical field normal component jump on the interior boundary.

The theory of the differential forms allows to produce weak solutions or variational analogs for Maxwell's equations with tensor coefficients [1,5]. The variational formulation written in terms of differential forms for modeling of 3D electromagnetic field in anisotropic medium is done in [5] for general case. This approach allows to write equations and do error estimates regardless to the choice of coordinate system. The Hodge operator included into material relation is responsible for metric properties of the medium as well as the anisotropic ones [2]. In discrete case, the Hodge operator is a Hodge matrix. The elements of the Hodge matrix contain the coefficients of the medium. However, the modeling results for such formulations with dense tensor in the 3D domain do not exist. The authors consider only diagonal tensors [9,19,10] on the structured brick triangulations.

The basic functions are associated with edges of the triangulation for vector FEM. The choice of the type of the triangulation is an important moment in the modeling in the media with isotropic and anisotropic properties. The local mass matrices are the discrete analogs of the Hodge matrices. If you choose brick triangulation, all the basic functions are

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collinear to system axes and the main axes of the tensor. One can get the correspondence between tensor elements and basic functions.

The orientation of the basic functions relative to system axes is arbitrary for tetrahedron triangulations. The correspondence between tensor elements and basic functions are becoming more complicated. So it is necessary to compute the projections of the tensor to the edges of the triangulation.

In this paper we present the computational scheme done on the base of vector FEM for the modeling of the 3D electromagnetic field by the local source in the medium with the anisotropic conductivity. The scheme was verified by the series of the computations on embedded structured triangulations. Mathematical modeling for the estimation of the dense tensor influence to the 3D electric field is done on the nonstructured tetrahedron triangulations. We consider some models of the anisotropic medium: the layered medium with diagonal and dense tensors and the different anisotropic objects in the isotropic half-space.

The structure of the work is the following one. In Section 2 the variational formulation for Helmholtz equation in arbitrary anisotropic medium is given. In Section 3, we present numerical estimates for verification the computational scheme. Section 4 shows the numerical results of electric field computation for layered model with transversally-isotropic layer. The influence of inclined anisotropic layer for electric field is investigated in Section 5. The problems with local object are considered in Sections 6 and 7.

## 2. Variational formulation

The electric field in the frequency domain is described by Helmholtz equation:

$$\begin{aligned}&\operatorname{rot}\mu^{-1}\operatorname{rot}\mathbf{E}+\big(iw\sigma-\varepsilon w^2\big)\mathbf{E}=-iw\mathbf{J},\\&\mathbf{E}\times\mathbf{n}|_{\partial\Omega e}=\mathbf{E}_0,\quad \mu^{-1}\operatorname{rot}\mathbf{E}\times\mathbf{n}|_{\partial\Omega h}=0,\end{aligned}\tag{1}$$

where $\mathbf{E}$ – electric field (V/m), $\mathbf{J}$ – current density in the source (A/m$^2$), $\mu$ – permeability (H/m), $\varepsilon$ – permittivity (F/m), $w$ – frequency (Hz), $\sigma$ – conductivity tensor of second rank (S/m), $\Omega$ – computational domain, $\partial\Omega e$ – the part of exterior domain boundary with electric boundary conditions, $\partial\Omega h$ – the part of exterior domain boundary with magnetic boundary conditions.

Let us introduce the functional spaces where we will seek the solution:

$$\begin{aligned}&H(\operatorname{rot};\Omega)=\big\{\mathbf{v}\in\big[L^2(\Omega)\big]^3:\operatorname{rot}\mathbf{v}\in\big[L^2(\Omega)\big]^3\big\}\\&H_0(\operatorname{rot};\Omega)=\big\{\mathbf{v}\in H^0(\operatorname{rot};\Omega):\mathbf{v}\times\mathbf{n}|_\Gamma=0\big\}\end{aligned}$$

where $\mathbf{n}$ – vector of outward normal to the boundary $\Gamma$ in domain $\Omega$. The norm in the space $H(\operatorname{rot};\Omega)$ is the following:

$$\|\mathbf{u}\|^2_{H(\operatorname{rot};\Omega)}=\int_\Omega\mathbf{u}\cdot\mathbf{u}\,d\Omega+\int_\Omega\operatorname{rot}\mathbf{u}\cdot\operatorname{rot}\mathbf{u}\,d\Omega$$

The variational formulation has the following form: *For given current density* $\mathbf{J}\in\mathbf{L}^2(\Omega)$ *such that* $\operatorname{div}\mathbf{J}=0$, *find* $\mathbf{E}\in H_0(\operatorname{rot};\Omega)$ *such that* $\forall\mathbf{W}\in H_0(\operatorname{rot};\Omega)$ the following equation is true.

$$\int_\Omega\mu^{-1}\operatorname{rot}\mathbf{E}\cdot\operatorname{rot}\mathbf{W}d\Omega-\int_\Omega\varepsilon w^2\mathbf{E}\cdot\mathbf{W}d\Omega+\int_\Omega iw\sigma\mathbf{E}\cdot\mathbf{W}d\Omega+\int_{\partial\Omega}\big(\mu^{-1}\operatorname{rot}\mathbf{E}\times\mathbf{n}\big)\cdot\mathbf{W}d\Omega=-\int_\Omega iw\mathbf{J}\cdot\mathbf{W}d\Omega\tag{2}$$

Denote $W^h$ finite-element subspace of curl conforming Nedelec's function [12,13]

$$W^h\subset\big\{\mathbf{v}\in H_0(\operatorname{rot};\Omega)\big\}$$

Nedelec's basic functions

$$W_i=\lambda_k\nabla\lambda_m-\lambda_m\nabla\lambda_k,\tag{3}$$

are associated with edges of nonstructured tetrahedron mesh. $\lambda_k$, $\lambda_m$ are baricentric coordinates of the edge $i$. Approximation $\operatorname{rot}\mathbf{E}$ has the same importance as the approximation of the field for the electromagnetic problem. It is known that the field $\mathbf{E}$ can be presented as a sum of two fields: solenoidal and potential. The basic function (3) describes solenoidal field. To obtain compete basis it is necessary to add new basic functions to the basis (3). This new functions should be the gradients of scalar functions and describe potential part of the field [21]. The additional functions to basis (3) has the following form:

$$W_j=\lambda_k\nabla\lambda_m+\lambda_m\nabla\lambda_k\tag{4}$$

For the complete first order basis there are two basic functions on each edge: one function (3) and another function (4).