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Application of transparent boundary conditions to high-order finite-difference schemes for the wave equation in waveguides

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ABSTRACT

We propose a method for generating finite-difference approximations of transparent boundary conditions (TBCs) with the fourth and sixth order in space. It is based on the wave equation solution continuation extra two or three layers of grid points outside the computational domain to use them in central-difference operators on approaching the boundary. We present the theoretical background of the method, give estimates of computational resources, and discuss accuracy and stability results of numerical tests in 1D and 2D cases.

0. Introduction

Explicit finite-difference schemes are a conventional technique for solving the wave equation. For unbounded domains, one needs to close a finite computational domain by open boundaries that imitate transparency for outgoing waves. There are several approaches based on using external domain Green's functions for generating such kind of conditions. Obviously, these approaches occupy a special place among the others, e.g. with perfectly matched layers (PML) [3], because they do not distort the original governing equations outside the open boundaries, i.e., they generate exact artificial boundary conditions. For instance, V.S. Ryaben'kii and coworkers developed theory and applications of difference potentials for external problems to efficiently implement correspondent difference non-reflecting boundary conditions, see [9,10] and references therein. Another well-known approach consists of deriving and approximation of pseudo-differential operators forming *transparent boundary conditions* (TBCs), see [11,12,6,1,13,2,14]. These conditions are non-local in both space and time; therefore, their use in numerical methods can be difficult. However, there are approaches considered in the cited papers that permit closing of the explicit time integration of the wave equation in the computational domain by a procedure of approximating TBCs at the open boundaries and obtaining stable and efficient *second-order accurate* schemes in both space and time.

Application of *high-order accurate* spatial schemes for the wave equation entails the need for corresponding matching accuracy at the open boundary. We propose an approach to increase the order of approximation of TBCs. As an example, we consider the wave propagation problem in a rectangular waveguide semi-infinite in the z direction: $-Z \le z < \infty$ for a $Z > 0$. Our computational domain is a parallelepiped with $-Z \le z \le 0$, the head portion of the waveguide, with the open boundary at $z = 0$ where we formulate and approximate TBCs with up to 6th spatial order of accuracy.

Let us note that standard PML approaches may lead to large numerical errors for wave propagation problems in waveguides; see analysis [5,4] of PML approximation issues in presence of physical boundaries.

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In the paper, we introduce an algorithm of high-order accuracy approximating TBCs and demonstrate that this sophisticated method can be treated by a sequence of sufficiently simple steps. The structure of the paper is as follows. Formulation of the problem is given in Section 1. Further, in Section 2, the TBCs are derived. Conventional discretization of the wave problem by difference schemes having $O(\tau^2 + h^{2p})$ order, $p = 1, 2, 3$, is described in Section 3. Section 4 is devoted to discretization of TBCs and estimation of their computational costs. In particular, it is shown that application of the approximate operator of the TBCs is similar (in the sense of computational expenses) to the integration of the wave equation in some additional grid layer with a fixed number of grid points in the z direction. In Section 5, the results of calculations for 1D and 2D test tasks illustrating the claimed accuracy $O(\tau^2 + h^{2p})$ of the difference schemes with TBCs and stability properties for large simulation times are given.

1. Problem formulation

Let us consider, for some $Z > 0$, $X > 0$, $Y > 0$, in a semi-infinite waveguide

$$\Omega = \{-Z \le z < \infty, 0 \le x \le X, 0 \le y \le Y\},$$

the following initial boundary value problem (IBVP) for a function $w \equiv w(t, x, y, z)$:

$$\begin{cases} w_{tt} - c^2(w_{xx} + w_{yy} + w_{zz}) = S(t, x, y, z), \quad (x, y, z) \in \Omega,\ t > 0 \\ w|_{z=-Z} = 0, \qquad \left.\dfrac{\partial w}{\partial n}\right|_{\Gamma} = 0 \\ w|_{t=0} = W_0(x, y, z), \qquad w_t|_{t=0} = W_1(x, y, z), \quad (x, y, z) \in \Omega. \end{cases} \tag{1}$$

Here $\Gamma = \partial\Omega \backslash \{z = -Z\}$ is the side waveguide boundary; n is the external normal; $c(x, y, z)$ is a sufficiently smooth function of the wave propagation speed; $S(t, x, y, z)$ is a sufficiently smooth source function; $W_0(x, y, z)$, $W_1(x, y, z)$ are sufficiently smooth initial data functions matching the boundary conditions of (1). We suppose that $S(t, x, y, z) = 0$, $W_0(x, y, z) = 0$, $W_1(x, y, z) = 0$, and $c(x, y, z) = \text{const}$ at $z \ge 0$.

Let us cut off the infinitely long portion of the waveguide from the head portion $-Z \le z \le 0$ and consider this bounded domain $\Omega_1 = \Omega \cap \{z \le 0\}$ with the side boundary $\Gamma_1 = \Gamma \cap \{z < 0\}$ and the open boundary $\Gamma_2 = \{0 \le x \le X, 0 \le y \le Y, z = 0\}$.

Parallel to (1), we consider a similar IBVP in the bounded domain Ω_1 for a function $v \equiv v(t, x, y, z)$:

$$\begin{cases} v_{tt} - c^2(v_{xx} + v_{yy} + v_{zz}) = S(t, x, y, z), \quad (x, y, z) \in \Omega_1,\ t > 0 \\ v|_{z=-Z} = 0, \qquad \left.\dfrac{\partial v}{\partial n}\right|_{\Gamma_1} = 0 \\ \mathcal{T} v|_{\Gamma_2} = 0 \\ v|_{t=0} = W_0(x, y, z), \qquad v_t|_{t=0} = W_1(x, y, z), \quad (x, y, z) \in \Omega_1, \end{cases} \tag{2}$$

where $\mathcal{T}$ denotes the TBCs operator; we derive it in Section 2.

The TBCs operator is defined as an operator providing the same solutions of both above IBVPs in Ω_1 for any admissible functions $S(t, x, y, z)$, $W_0(x, y, z)$, and $W_1(x, y, z)$.

2. Transparent boundary conditions

We apply the following scheme for deriving the TBCs operator [11,12]. First we write down the explicit formulas of solution continuation into the truncated portion of the waveguide, and then by passing to the limit, we obtain $\mathcal{T}$ on the separating boundary. The solution continuation formulas will be used while generating a discrete procedure that provides transparency of boundary Γ_2 for high-order finite-difference schemes. In other words the proposed approximation of $\mathcal{T}$ will be indirect. Note that in the conventional second-order accurate difference scheme at the open boundary, a direct approximation of $\mathcal{T}$ is usually used [2].

Let us consider the following auxiliary IBVP in the semi-infinite waveguide $\Omega_2 = \{0 \le z < \infty, 0 \le x \le X, 0 \le y \le Y\}$:

$$\begin{cases} u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0, \quad (x, y, z) \in \Omega_2,\ t > 0 \\ u|_{z=0} = f(t, x, y), \quad t > 0 \\ \left.\dfrac{\partial u}{\partial n}\right|_{\Gamma \cap \{z \ge 0\}} = 0 \\ u|_{t=0} = 0, \qquad u_t|_{t=0} = 0, \quad (x, y, z) \in \Omega_2 \end{cases} \tag{3}$$

where $c = \text{const} > 0$, $f(t, x, y)$ is a function with zero normal derivative at $\Gamma \cap \{z = 0\}$. This IBVP parameterizes solutions to the wave equation in Ω_2 in terms of $f(t, x, y)$. To solve (3) by the Fourier method, the solution $u(t, x, y, z)$ is represented in terms of the harmonics $\varphi_{\alpha,\beta}(x, y) = \cos(\pi\alpha x/X)\cos(\pi\beta y/Y)$, $\alpha, \beta = 0, \ldots, \infty$: