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An algorithm of the method of difference potentials for domains with cuts

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ABSTRACT

The method of Difference Potentials (DPM) is applied to solving a Dirichlet problem for the Laplace equation in a square with a cut. The DPM approach has been modified to achieve a more efficient numerical algorithm with respect to computational time. The considered problem can be a prototype for other problems formulated in domains with cuts including elastic problems related to cracks.

1. Introduction

The method of Difference Potentials (DPM) with projections was first suggested by Ryaben'kii in 1969 in his Habilitation dissertation (see [7] and [8]), and was significantly modified afterwards in numerous publications. One of the key directions of development and applications of DPM is related to solving linear boundary value problems (BVPs). In the current paper the DPM is applied to solving BVP with a cut. The generic scheme of the method contains three major steps. The general solution of the homogeneous difference equation is presented via a function that depends on an arbitrary function from some auxiliary space. The auxiliary space is called the space of densities. In turn, the general solution of the homogeneous equation in a domain with a cut is called a potential with the density from the auxiliary space. At the next step the density is found which satisfies the boundary conditions. Finally, the solution to the BVP is reduced to calculation of the difference potential with the density evaluated at the previous step. It goes without saying each step affects the efficiency of the algorithm. The class of difference potentials with the density from the space of the so-called clear traces was first introduced in [5]. Such a difference potential proves to be fully determined by its clear trace. A criteria of the clear trace suggested in [5] allows the calculation of the potential to be reduced to solving an auxiliary BVP. The choice of the auxiliary problem is quite flexible.

For the first time the problem in question was tackled by Kamenetskii and Ryaben'kii in [3]. The potential constructed there was further extended in [2,9,10] for solving the gas dynamic equations in the hodograph plane for design of turbine blades. It should be noted that in [3] the dimension of space of densities exceeds the dimension of space of solutions as much as twice. As shown in [4], one can minimize the dimension of the space of clear traces. Thereby, the number of unknown variables describing the density can also be minimized. As is proven in [6], piecewise regular solutions of a homogeneous equation can be represented by the potentials with the density from the space of so-called jumps. This space

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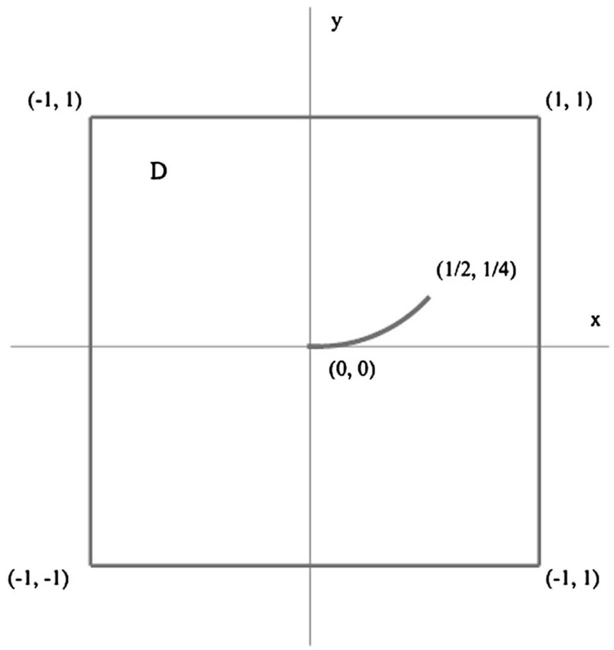

Fig. 1. Domain with a cut.

proved to be related to the space of minimal clear traces for such problems. It should be noted that the solution of the BVP in domains with cuts is not always represented by piecewise regular functions. However, as shown in the current paper, the solution to this problem can be obtained via a potential with the density from the space of jumps that has the minimal dimension. Therefore, the suggested algorithm is more efficient than that of [3].

The current paper is organized as follows. In Section 2, the general formulation of the problem is provided. A BVP with Dirichlet boundary conditions is considered in a square with a cut. In the next section, this BVP is formulated in a finite-difference form. The section also contains all basic definitions of the DPM formalism. Section 4 is devoted to the construction of an efficient numerical algorithm for solving the discrete BVP. The section is followed by the Conclusion.

2. General formulation of the problem

Consider square $D = \{(x, y) \mid |x| \leqslant 1, |y| \leqslant 1\}$ on plane $x0y$. In D introduce cut Γ defined by equation $y = x^2$, $0 \leqslant x \leqslant \frac{1}{2}$ (Fig. 1). Thus, the cut starts at point $A(0, 0)$ and ends at point $B(1/2, 1/4)$. Next, we identify two sides of the cut Γ: Γ_{left} and Γ_{right}.

Introduce linear space V_D of functions smooth enough outside Γ such that they are determined on $\overline{D}$, have a double value at Γ and equal to zero at the boundary ∂D of the domain D.

Then, consider boundary value problem (BVP):

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \quad \text{if } (x, y) \in \overline{D} \tag{1}$$

$$v_{|\partial D} = 0_{|\partial D} \tag{2}$$

$$v_{|\Gamma_{left}} = \phi_{left}(x), \quad 0 \leqslant x \leqslant \frac{1}{2} \tag{3}$$

$$v_{|\Gamma_{right}} = \phi_{right}(x), \quad 0 \leqslant x \leqslant \frac{1}{2} \tag{4}$$

Assume that ϕ_{left} and ϕ_{right} are prescribed functions that satisfy the following consistency conditions at the ends of the cut: $\phi_{left}(0) = \phi_{right}(0)$, $\phi_{left}(\frac{1}{2}) = \phi_{right}(\frac{1}{2})$.

3. Finite-difference counterpart of the problem formulation

Let us introduce a grid in D and formulate a counterpart of governing equation (1) with boundary conditions (3), (4) at the boundary ∂D (2) and cut Γ.