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A two-grid method for elliptic problem with boundary layers

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ABSTRACT

A two-grid method for the elliptic equation with a small parameter ε multiplying the highest derivative is investigated. The difference schemes with the property of ε-uniform convergence on a uniform mesh and on Shishkin mesh are considered. In both cases, a two-grid method for resolving the difference scheme is investigated. A two-grid method has features that are concerned with a uniform convergence of a difference scheme. To increase the accuracy, the Richardson extrapolation in two-grid method is applied. Numerical results are discussed.

Introduction

The construction of uniformly accurate finite difference methods for singularly perturbed boundary value problems is actual problem of numerical analysis. The application of classical difference schemes for a singularly perturbed problem leads to large errors for small values of perturbing parameter. The uniform convergence of a difference scheme for such problem can be provided by fitting the scheme to a boundary layer component [8,14] or by using a mesh which is dense in a boundary layer [3,21].

We consider the two-dimensional linear elliptic problem with regular boundary layers. The difference scheme for such a problem is a system of linear equations which can be solved by the iterative method. It is known that the application of multigrid methods leads to essential reduction of the number of arithmetical operations. Multigrid methods were developed in [4,5,7,9,12,18,20,24] and in other works.

Multigrid methods for convection–diffusion elliptic problems were investigated in [10,11,17,24] and in other works. It is known that difficulties in the resolution of the associated linear systems can appear, because the matrix is unsymmetric and have a large condition number for small values of a parameter ε [10].

In [17], multigrid method for linear elliptic convection–diffusion problem is investigated. The method of finite elements is used for the discretization of differential problem. The question of ε-uniform convergence of the difference scheme is not considered. Block-structured preconditioning approach with estimation of the convergence rate independent in ε and h is proposed.

In [15], linear elliptic reaction–diffusion problem is solved by multigrid method. Bakhvalov and Shishkin meshes [3,21] are used to achieve ε-uniform convergence of the difference scheme. Authors show that standard direct solvers exhibit poor scaling behavior, with respect to the parameter ε, when solving the resulting linear systems. A robust for small values of ε preconditioning approach is proposed.

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In [6,23,24] and in other works, multigrid methods are applied for modeling flows of viscous liquid. Computationally these problems are rather complex problems. The difference schemes with provable ε-uniform convergence for Navier–Stokes equations are absent so far. Here parameter ε is inversely proportional to Reynolds number.

Two-grid methods as a special case of multigrid methods were investigated in [2,28] and in other works. According to the conception of the two-grid algorithm, if the difference scheme is resolved on the basis of iterations then at first the problem is solved on a coarse mesh. Secondly, the new mesh solution is interpolated to nodes of the fine mesh and is used as the initial approximation for following iterations. It leads to reduction of the number of iterations on the fine mesh and also reduces the number of arithmetical operations.

The application of two-grid methods to singularly perturbed boundary value problems is an interesting subject for investigations. This question was considered in [25,26], where a two-grid method was applied for solving a singularly perturbed boundary value problem in the case of the second order nonlinear ordinary differential equation. In [27], two-grid method on uniform grids was investigated for nonlinear singularly perturbed elliptic problem. In [1], a two-grid method was investigated in the case of reaction–diffusion elliptic equation on Bakhvalov and Shishkin meshes.

A goal of this article is the investigation of two-grid method for the elliptic problem with regular boundary layers.

The main results of our article are the following.

1. We offer to apply two-grid method to resolve ε-uniform convergence difference schemes for linear elliptic problem with regular boundary layers on uniform and on Shishkin meshes, using ε-uniform interpolation formulas and Richardson extrapolation.
2. In the case of the scheme on a uniform mesh we propose to use constructed nonpolynomial interpolation formula, which is exact on boundary layer components, in two-grid method. We show the advantage of this formula in numerical experiments.
3. We prove that the formula of the linear interpolation on Shishkin mesh has ε-uniform accuracy and can be used successfully in two-grid method. To increase ε-uniform accuracy of the difference scheme on Shishkin mesh without additional calculations, we propose to use Richardson extrapolation formula in two-grid method.

Earlier we obtained that the uniform accuracy of an interpolation formula for a function with a boundary layer component can be achieved by the fitting to a boundary layer component [27,29,31] and by using a mesh which is dense in a boundary layer [30].

Richardson extrapolation method to increase ε-uniform accuracy of the difference schemes for a singularly perturbed problems was investigated in [16,22] and in other works. We investigate the possibility to apply Richardson extrapolation in a two-grid method using known solutions of the difference scheme on both meshes.

Notation. Everywhere C and C_j, $j \geqslant 0$ mean the positive constants that are independent on a parameter ε and the mesh steps. Let $[v]_\Omega$ be the projection of a function $v(x,y)$ on the mesh Ω. Let us define norms for continuous functions and the norm for a mesh function:

$$\|v(x)\| = \max_{0\leqslant x\leqslant 1}|v(x)|, \qquad \|v(x,y)\| = \max_{0\leqslant x,y\leqslant 1}|v(x,y)|, \qquad \|v^h\|_h = \max_{i,j}|v^h_{i,j}|.$$

1. Differential problem

We consider the following boundary value problem:

$$\begin{aligned} &\varepsilon u_{xx} + \varepsilon u_{yy} + a(x)u_x + b(y)u_y - c(x,y)u = f(x,y), \quad (x,y)\in\Omega, \\ &u(x,y) = g(x,y), \quad (x,y)\in\Gamma, \end{aligned} \tag{1.1}$$

where $\Omega = (0,1)^2$, $\Gamma = \overline{\Omega}\setminus\Omega$, functions a, b, c, f, g are sufficiently smooth,

$$a(x) \geqslant \alpha > 0, \qquad b(y) \geqslant \beta > 0, \qquad c(x,y) \geqslant 0, \qquad \varepsilon > 0. \tag{1.2}$$

It is known that the solution of the problem (1.1) is uniformly bounded and according to [19] it is possible to extract exponential boundary layer components:

$$\begin{aligned} u(x,y) = {}&\big[g(0,y) - w(0,y)\big]\Phi(x) + \big[g(x,0) - w(x,0)\big]\Theta(y) \\ &- \big[g(0,0) - w(0,0)\big]\Phi(x)\Theta(y) + p(x,y), \quad p(x,y) - w(x,y) = O(\varepsilon), \end{aligned} \tag{1.3}$$

where $p(x,y)$ is a regular component with bounded first derivatives:

$$\big|p'_x(x,y)\big|, \qquad \big|p'_y(x,y)\big| \leqslant C_0,$$

and $w(x,y)$ is the solution of the degenerate problem: