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Multiple time-dependent coefficient identification thermal problems with a free boundary

M.S. Hussein [a,b], D. Lesnic [a,*], M.I. Ivanchov [c], H.A. Snitko [d]

[a] *Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK*
[b] *Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq*
[c] *Faculty of Mechanics and Mathematics, Department of Differential Equations, Ivan Franko National University of Lviv, 1, Universytetska str., Lviv, 79000, Ukraine*
[d] *Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of National Academy of Science of Ukraine, 3-b, Naukova Str., Lviv, 79060, Ukraine*

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Multiple time-dependent coefficient identification thermal problems with an unknown free boundary are investigated. The difficulty in solving such inverse and ill-posed free boundary problems is amplified by the fact that several quantities of physical interest (conduction, convection/advection and reaction coefficients) have to be simultaneously identified. The additional measurements which render a unique solution are given by the heat moments of various orders together with a Stefan boundary condition on the unknown moving boundary. Existence and uniqueness theorems are provided. The nonlinear and ill-posed problems are numerically discretised using the finite-difference method and the resulting system of equations is solved numerically using the MATLAB toolbox routine *lsqnonlin* applied to minimizing the nonlinear Tikhonov regularization functional subject to simple physical bounds on the variables. Numerically obtained results from some typical test examples are presented and discussed in order to illustrate the efficiency of the computational methodology adopted.

1. Introduction

Inverse coefficient identification problems (ICIP) for partial differential equations are some of the most complicated and practically important problems. Being in addition nonlinear, optimization techniques are mainly used for their numerical solutions, as well as various modifications tailored to the properties of the corresponding direct problems (monotonicity or/and smoothness of their solutions, etc.), [13]. ICIP's with one or several unknown coefficients play a substantial role in the theory and application of inverse problems. A great attention was paid to this kind of inverse problems due to the industrial applications in practice, for instance, the determination of the thermal conductivity, heat capacity, absorption coefficient, etc., in the field of heat conduction or porous media.

Many practical problems involve a free boundary and the Stefan problem is a typical example of a problem of this kind, [22,23]. Under suitable changes of variables, free boundary problems can be reduced to ICIP's in a fixed domain.

* Corresponding author.
E-mail addresses: mmmsh@leeds.ac.uk (M.S. Hussein), amt5ld@maths.leeds.ac.uk (D. Lesnic), ivanchov@franko.lviv.ua (M.I. Ivanchov), snitkog@ukr.net (H.A. Snitko).

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This approach opens the new and more complicated area of inverse problems that combine free boundary problems with coefficient identification problems. One of the main feature of these problems is that the unknowns depend solely on the time variable and this enables a neat mathematical treatment based on Green's functions, [10].

Prior to this study, references [4,7,9] investigated both theoretically and numerically several such combined formulations for the retrieval of the free boundary together with the thermal diffusivity both which are unknown time-dependent functions. The theoretical investigation has been extended recently to the case of several multiple coefficients in [18,19] and it the purpose of this study to, apart from some theoretical clarifications which are elaborated in Section 2, perform the numerical realization using the finite-difference method (FDM) combined with a nonlinear least-squares toolbox MATLAB routine, see Sections 3 and 4. In Section 5, we provide numerical results and discussion, whilst Section 6 presents an extension to a triple unknown coefficient identification. Finally, conclusions are highlighted in Section 7.

2. Mathematical formulation

Consider the one-dimensional time-dependent heat equation

$$\frac{\partial u}{\partial t}(x,t) = a(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) + b(t)\frac{\partial u}{\partial x}(x,t) + c(t)u(x,t) + f(x,t), \quad (x,t) \in \Omega \tag{1}$$

for the unknown temperature $u(x,t)$ in the domain $\Omega = \{(x,t)|\ 0 < x < h(t), 0 < t < T < \infty\}$ with unknown free smooth boundary $x = h(t) > 0$ and time-dependent coefficients $b(t)$ and $c(t)$ representing the convection/advection and reaction coefficients, respectively. Also in (1), $f(x,t)$ represents a given heat source, whilst $a(x,t) > 0$ is the given thermal diffusivity. In many applications, [4,8,19,24], the thermal diffusivity depends on time only, but here we envisage a more general physical situation in which the thermal conductivity depends on time and the heat capacity depends on space such that their ratio defined as the thermal diffusivity depends on both space and time. To give more physical meaning to the inverse problem, we have in mind a process in which a finite slab is undertaking radioactive decay such that its diffusivity, convection and reaction coefficients are unknown but they depend on time [1, Chap. 13], [16]. We finally mention that extensions to cases when the time-dependent heat source is also unknown or when some unknown coefficients may depend on space as well have recently been considered elsewhere, [5,6].

The initial condition is

$$u(x,0) = \phi(x), \quad 0 \le x \le h(0) =: h_0, \tag{2}$$

where $h_0 > 0$ is given, and the Dirichlet boundary conditions are

$$u(0,t) = \mu_1(t), \quad u(h(t),t) = \mu_2(t), \quad t \in [0,T]. \tag{3}$$

As over-determination conditions we consider, [18],

$$h'(t) + u_x(h(t),t) = \mu_3(t), \quad t \in [0,T], \tag{4}$$

$$\int_0^{h(t)} u(x,t)dx = \mu_4(t), \quad t \in [0,T], \tag{5}$$

$$\int_0^{h(t)} xu(x,t)dx = \mu_5(t), \quad t \in [0,T]. \tag{6}$$

Note that $\mu_4(t)$ and $\mu_5(t)$ represent the specification of the energy or, mass of the heat conducting system and heat momentum, respectively, [2,11,15]. Also, equation (4) represents a Stefan interface moving boundary condition.

Now we perform the change of variable $y = x/h(t)$ to reduce the problem (1)–(6) to the following inverse problem for the unknowns $h(t)$, $b(t)$, $c(t)$ and $v(y,t) := u(yh(t),t)$:

$$\frac{\partial v}{\partial t}(y,t) = \frac{a(yh(t),t)}{h^2(t)}\frac{\partial^2 v}{\partial y^2}(y,t) + \frac{b(t) + yh'(t)}{h(t)}\frac{\partial v}{\partial y}(y,t) + c(t)v(y,t) + f(yh(t),t), \quad (y,t) \in Q_T \tag{7}$$

in the fixed domain $Q_T := \{(y,t) : 0 < y < 1, 0 < t < T\} = (0,1) \times (0,T)$,

$$v(y,0) = \phi(h_0 y), \quad y \in [0,1], \tag{8}$$

$$v(0,t) = \mu_1(t), \quad v(1,t) = \mu_2(t), \quad t \in [0,T], \tag{9}$$

$$h'(t) + \frac{1}{h(t)}v_y(1,t) = \mu_3(t), \quad t \in [0,T], \tag{10}$$

$$h(t)\int_0^1 v(y,t)dy = \mu_4(t), \quad t \in [0,T], \tag{11}$$