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Error estimates for the moving least-square approximation and the element-free Galerkin method in n-dimensional spaces

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A R T I C L E   I N F O

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A B S T R A C T

The moving least square (MLS) approximation is one of the most important methods to construct approximation functions in meshless methods. For the error analysis of the MLS-based meshless methods it is fundamental to have error estimates of the MLS approximation in the generic n-dimensional Sobolev spaces. In this paper, error estimates for the MLS approximation are obtained in the $W^{k,p}$ norm in arbitrary n dimensions when weight functions satisfy certain conditions. The element-free Galerkin (EFG) method is a typical Galerkin method combined with the use of the MLS approximation. The error results of the MLS approximation are then used to yield error estimates of the EFG method for solving both Neumann and Dirichlet boundary value problems. Finally, some numerical examples are given to confirm the theoretical analysis.

1. Introduction

In the past several decades, finite difference methods, finite element methods (FEMs) and boundary element methods (BEMs) are dominant computational methods for the numerical solution of boundary value problems. These traditional numerical methods depend on the generation of meshes, adapted or not. Mesh generation in some situations is still arduous, time consuming and fraught with pitfalls. Meshless (or meshfree) methods, which are approximations based on nodes, can overcome the disadvantage that traditional numerical methods depend on the mesh of the solution domain.

The moving least-square (MLS) [18] is an approximation method developed by Lancaster and Salkauskas for surface generation problems. In this method, continuous functions are generated from a cluster of unorganized sampled point values by computing a weighted least squares approximation. The main difference between traditional numerical methods and meshless methods is the way in which the shape function is constructed. The MLS approximation is a well developed method used for constructing meshless shape functions, since it starts from scattered nodes instead of elements or meshes.

In 1992, by combining the MLS approximation and the global Galerkin weak form, Nayroles et al. pioneered the diffuse element method (DEM) for solving boundary value problems [33]. In 1994, Belytschko et al. modified the DEM and developed the element-free Galerkin (EFG) method [7]. As stated in [5], a numerical method for the solution of a parameter-dependent problem is said to exhibit locking if the solution deteriorates as the parameter tends to a limiting value. It is well known that various FEMs suffer from the volumetric locking, and thus special treatments are needed to overcome the locking phenomenon [5,43]. However, the results in [7,42] show that the EFG method can avoid volumetric locking and has

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high computational precision and stability. Therefore, the EFG method becomes one of the most important meshless methods. In the past two decades, the EFG method has been achieved remarkable progress in solving a broad class of science and engineering problems. After the EFG method, the research into meshless methods became very active and lots of meshless methods have been developed using the MLS approximation, such as the *h*-*p* meshless method [14], the finite point method (FPM) [34], the meshless local Petrov–Galerkin (MLPG) method [4,12], the meshless local weak–strong (MLWS) method [13], the meshless local boundary integral equation (BIE) method [4], the boundary node method (BNM) [31], the symmetric Galerkin BNM [24,20], the dual BNM [22], and so on.

Because shape functions are the basis of meshless methods, for the error analysis of the MLS-based meshless methods it is sufficient to have error estimates of the MLS approximation in the generic Sobolev spaces. For the approximation of a continuous function and its first and second order derivatives, Levin [19], Armentano and Durán [2], Armentano [1] and Zuppa [44] established error estimates for the MLS approximation in the L^∞ norm. It is well-known that in many cases the function to be approximated is less regular, thus it is certainly important to obtain error estimates in Sobolev spaces under weaker regularity assumptions.

With some assumptions on weight functions, Armentano [1] and Zuppa [45] further established error estimates in L^2 and $W^{2,p}$, respectively. Nevertheless, they do not obtain error estimates in the generic Sobolev spaces $W^{k,p}$, which is more suitable for error analysis of the associated meshless methods.

With some assumptions on MLS shape functions, Cheng and Cheng [9] established error estimates in $W^{k,p}$. Their assumptions appear somewhat restrictive, since the explicit formulation of MLS shape functions is unavailable for the generic case.

Recently, under appropriate assumptions on weight functions, we established error estimates for the MLS approximation, and then obtained error analysis and convergence study of the Galerkin BNM in Sobolev spaces [24,20]. More recently, Ren et al. [35] established error estimates for the MLS approximation when weight functions satisfy certain conditions and the quadratic polynomial basis is used. The assumption on weight functions can be fulfilled for a variety of generally used weight functions. However, the results in [24,20] were only shown in the H^k norm for one- and two-dimensional problems, while the results in [35] were only shown in $W^{k,p}$ in two dimensions.

In addition, some MLS variants and their error estimates have also been presented. Liu et al. [27] established an MLS reproducing kernel method and its error. Duarte and Oden [14] and Hu [16] obtained error estimates for the *h*-*p* clouds methods. Wang et al. [41] given error analysis of an interpolating MLS. Salehi and Dehghan developed an MLS reproducing polynomial meshless method [36] and a generalized MLS reproducing kernel method [37] and analyzed the associated errors. Mirzaei et al. [30] discussed error bounds of a generalized MLS method. Based on the generalized MLS method, a direct MLPG method has been developed by Mirzaei and Schaback for elliptic interface problems [39]. Besides, Refs. [29,3] give error estimates of MLS-based meshless collocation methods for linear and nonlinear integral equations.

In this paper, one goal is to derive error estimates for the MLS approximation in the generic $W^{k,p}$ norm in arbitrary n dimensions when weight functions satisfy some conditions. Following the suggestion of a reviewer, we finally point out that Mirzaei [28] uses a shifted and scaled polynomial basis function to analysis the error of the MLS approximation in 2015. By using the commonly used and non-scaled polynomial basis functions, the error estimate in the present work is established in a different way.

Then, we use the error results of the MLS approximation to yield optimal order error estimates of the EFG method for solving Neumann boundary value problems. Additionally, by using the penalty method to treat Dirichlet boundary conditions, error estimates of the EFG method are also established for solving Dirichlet boundary value problems. Finally, some numerical examples are selected to confirm the theoretical analysis.

The rest of this paper is outlined as follows. Section 2 presents some notations to be used later. Then, a brief description of the MLS approximation is given in Section 3. Sections 4 and 5 present detailed error estimates for the MLS approximation and the EFG method, respectively. Numerical results are provided in Section 6. Section 7 contains conclusions.

2. Notations

Throughout the paper, the letter n is a positive integer and is used for the spatial dimension. Let $\Omega \subset \mathbb{R}^n$ be a nonempty, open bounded set with a Lipschitz continuous boundary Γ. A generic point in $\mathbb{R}^n$ is denoted by $\mathbf{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ or $\mathbf{y} = (y_1, y_2, \cdots, y_n)^{\mathrm{T}}$.

A multi-index is an order collection of n nonnegative integers, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)^{\mathrm{T}}$. For any α, the length of α is $|\alpha| = \sum_{i=1}^n \alpha_i$, and the α-th order partial derivative of a function $u(\mathbf{x})$ is $D^\alpha u(\mathbf{x}) = \partial^{|\alpha|} u(\mathbf{x}) / \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}$. As usual, $D^{\mathbf{0}} u(\mathbf{x}) = u(\mathbf{x})$.

For any $\mathbf{x} \in \bar{\Omega} = \Omega \cup \Gamma$, assume that the influence domain of $\mathbf{x}$ is $\Re(\mathbf{x})$ with radius $r(\mathbf{x})$, that is

$$\Re(\mathbf{x}) = \left\{ \mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y}\| \le r(\mathbf{x}) \right\}.$$

Let $\{\mathbf{x}_i\}_{i=1}^N$ be an arbitrarily chosen set of N nodes $\mathbf{x}_i \in \bar{\Omega}$. This set is used for defining a finite open covering $\{\Re_i\}_{i=1}^N$ of $\bar{\Omega}$ composed of N balls $\Re_i$ centered at $\mathbf{x}_i$, $i = 1, 2, \cdots, N$, where

$$\Re_i = \Re(\mathbf{x}_i) = \left\{ \mathbf{y} \in \mathbb{R}^n : \|\mathbf{x}_i - \mathbf{y}\| \le r_i \right\} \tag{2.1}$$