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A certified natural-norm successive constraint method for parametric inf–sup lower bounds

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ABSTRACT

We present a certified version of the Natural-Norm Successive Constraint Method (cNNSCM) for fast and accurate Inf–Sup lower bound evaluation of parametric operators. Successive Constraint Methods (SCM) are essential tools for the construction of a lower bound for the inf–sup stability constants which are required in *a posteriori* error analysis of reduced basis approximations. They utilize a Linear Program (LP) relaxation scheme incorporating continuity and stability constraints. The natural-norm approach *linearizes* a lower bound of the inf–sup constant as a function of the parameter. The Natural-Norm Successive Constraint Method (NNSCM) combines these two aspects. It uses a greedy algorithm to select SCM control points which adaptively construct an optimal decomposition of the parameter domain, and then apply the SCM on each domain.

Unfortunately, the NNSCM produces no guarantee for the quality of the lower bound. Through multiple rounds of optimal decomposition, the new cNNSCM provides an upper bound in addition to the lower bound and lets the user control the gap, thus the quality of the lower bound. The efficacy and accuracy of the new method is validated by numerical experiments.

1. Introduction

For affinely parametrized partial differential equations, the certified reduced basis method (RBM) [20,23,25,11] utilizes an Offline–Online computational decomposition strategy to produce surrogate solution (of dimension N) in a time that is of orders of magnitude shorter than what is needed by the underlying numerical solver of dimension $\mathcal{N} \gg N$ (called *truth* solver hereafter). The RBM relies on a projection into a low dimensional space spanned by truth approximations at an optimally sampled set of parameter values [2,9,21,22,18]. This low-dimensional manifold is generated by a greedy algorithm making use of a rigorous *a posteriori* error bounds for the field variable and associated functional outputs of interest which also guarantees the fidelity of the surrogate solution in approximating the truth approximation. The high efficiency and accuracy of RBM render it an ideal candidate for practical methods in the real-time and many-query contexts.

This crucial *a posteriori* error bound is residual-based and requires an estimate (lower bound) for the stability factor of the discrete partial differential operator, that is the coercivity or inf–sup constant. In the RBM context, given any parameter value this stability factor must be estimated efficiently. So it should also admit an Offline–Online structure for which the Online expense is independent of $\mathcal{N}$. Moreover, the optimality of the low-dimensional RB manifold is dependent on the

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quality of this estimate as a parameter-dependent function, so the lower bound should not be too pessimistic. There are several approaches in the literature. A Successive Constraint Method (SCM) is proposed in [14] and subsequently improved in [4,5,28,29]. It is a framework incorporating both continuity *and* stability information whose Online component is the resolution of a small-size Liner Programming (LP) problem. Hence, this procedure is rather efficient. However, the classical inf–sup formulation has couple of undesirable attributes – a Q^2-term affine parameter expansion (resulting from a squaring of the operator), and loss of (even local) concavity. On the other hand, a "natural-norm" method is proposed in [7,26]. Its linearized-in-parameter inf–sup formulation has several desirable approximation properties – a Q-term affine parameter expansion, and first order (in parameter) concavity; however, the lower bound procedure is rather crude – a framework which incorporates only continuity information. A natural-norm SCM approach is proposed in [13] combining the "linearized" inf–sup statement with the SCM lower bound procedure. The former (natural-norm) provides a smaller optimization problem which enjoys intrinsic lower bound properties. The latter (SCM) provides a systematic optimization framework: a Linear Program relaxation which readily incorporates effective stability constraints. The natural-norm SCM performs very well in particular in the Offline stage: it is typically an order of magnitude less costly than either the natural-norm or "classical" SCM approaches alone. However, unlike the classical SCM, it provides no upper-bound thus no control of the quality of the lower bound. This often results in extremely pessimistic estimate.

There are other methods such as the θ-methods that were compared to SCMs in [12]. In this paper, we focus on the SCM-type of approaches and propose a certified version of the NNSCM (cNNSCM). Without significantly degrading the efficiency, it provides an upper-bound and thus a mechanism for the user to easily control the quality of the lower bound. As a result, the lower bound of the new cNNSCM may be orders of magnitude more accurate than the original NNSCM thanks to the design of multiple rounds of decomposition of the parameter domain. The method is tested on two elliptic partial differential equations. In what follows, we use the same notation as in [13] and denote the classical SCM method [14,4,5] as SCM^2 in order to differentiate it from the new natural-norm type of approaches. Here, the (squared) superscript suggests the undesired Q^2-term affine parameter expansion in the classical method.

This paper is organized as follows. In Section 2, we review the background materials including the RBM, its *A Posteriori* error estimation and the involved stability constant. Section 3 describes the natural-norm SCM. The new certified NNSCM is proposed in Section 4. Numerical validations are presented in Section 5, and finally some concluding remarks are offered in Section 6.

2. Background

For the completeness of this paper and to put the concerned method into context, we introduce the necessary background materials in this section. To that end, this section covers the truth solver and the related stability constants, the reduced basis method, and the *A Posteriori* error estimate needed therein.

2.1. Notations

We use $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) to denote a bounded physical domain with boundary $\partial\Omega$. We introduce a closed parameter domain $\mathcal{D} \in \mathbb{R}^P$, a point ($P$-tuple) in which is denoted $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_P)$. A set of N parameter values will be differentiated by superscripts $\{\boldsymbol{\mu}^i\}_{i=1}^N$. Let us then define the Hilbert space X equipped with inner product $(\cdot,\cdot)_X$ and induced norm $\|\cdot\|_X$. Here $(H_0^1(\Omega))^{\mathcal{V}} \subset X \subset (H^1(\Omega))^{\mathcal{V}}$ ($\mathcal{V} = 1$ for a scalar field and $\mathcal{V} > 1$ for a vector field) [24,1]. We introduce a parametrized bilinear form. $a(\cdot,\cdot;\boldsymbol{\mu})$: $X \times X \to \mathbb{R}$ is such that

- it is inf–sup stable and continuous over X: $\beta(\boldsymbol{\mu}) > 0$ and $\gamma(\boldsymbol{\mu})$ is finite $\forall \boldsymbol{\mu} \in \mathcal{D}$, where

$$\beta(\boldsymbol{\mu}) = \inf_{w \in X} \sup_{v \in X} \frac{a(w, v; \boldsymbol{\mu})}{\|w\|_X \, \|v\|_X}, \text{ and } \gamma(\boldsymbol{\mu}) = \sup_{w \in X} \sup_{v \in X} \frac{a(w, v; \boldsymbol{\mu})}{\|w\|_X \, \|v\|_X};$$

- a is "affine" in the parameter: $a(w, v; \boldsymbol{\mu}) = \sum_{q=1}^{Q} \Theta_q(\boldsymbol{\mu}) a_q(w, v)$.

Finally, we introduce two linear bounded functionals $f(\cdot;\boldsymbol{\mu}) : X \to \mathbb{R}$ and $\ell(\cdot;\boldsymbol{\mu}) : X \to \mathbb{R}$ that are also affine in the parameter. The following continuous problem is then well-defined.

(P_C) Given $\boldsymbol{\mu} \in \mathcal{D}$, find $u(\boldsymbol{\mu}) \in X$ such that $a(u(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v, \boldsymbol{\mu}), \forall v \in X$.

For many applications, we concern a scalar quantity of interest as $s(\boldsymbol{\mu}) = \ell(u(\boldsymbol{\mu}), \boldsymbol{\mu})$. To discretize this problem, we consider for an example a finite element approximation space (of dimension $\mathcal{N}$) $X^{\mathcal{N}} \subset X$. Suppose that the discretized bilinear form remains inf–sup stable (and continuous) over $X^{\mathcal{N}}$ with constants $\beta^{\mathcal{N}}(\boldsymbol{\mu}) > 0$ and $\gamma^{\mathcal{N}}(\boldsymbol{\mu})$ being finite $\forall \boldsymbol{\mu} \in \mathcal{D}$, where

$$\beta^{\mathcal{N}}(\boldsymbol{\mu}) = \inf_{w \in X^{\mathcal{N}}} \sup_{v \in X^{\mathcal{N}}} \frac{a^{\mathcal{N}}(w, v; \boldsymbol{\mu})}{\|w\|_{X^{\mathcal{N}}} \, \|v\|_{X^{\mathcal{N}}}} \text{ and } \gamma^{\mathcal{N}}(\boldsymbol{\mu}) = \sup_{w \in X^{\mathcal{N}}} \sup_{v \in X^{\mathcal{N}}} \frac{a^{\mathcal{N}}(w, v; \boldsymbol{\mu})}{\|w\|_{X^{\mathcal{N}}} \, \|v\|_{X^{\mathcal{N}}}}.$$