Applied Numerical Mathematics 99 (2016) 109–120

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Solutions of linear second order initial value problems by using Bernoulli polynomials

Anna Napoli *

Department of Mathematics, University of Calabria, 87036 Rende (CS), Italy

ARTICLE INFO

Article history:
Received 1 November 2014
Received in revised form 24 June 2015
Accepted 6 August 2015
Available online 11 September 2015

Keywords:
Initial value problems
Bernoulli polynomials
Spectral methods

ABSTRACT

In this paper we use Bernoulli polynomials to derive a new spectral method to find the numerical solutions of second order linear initial value problems. Stability and error analysis of this method are studied. Numerical examples are presented which support theoretical results and provide favorable comparisons with other existing methods.

1. Introduction

Consider the second-order linear differential equation

$$y''(x) = f(x)y'(x) + g(x)y(x) + r(x), \qquad x \in I = [a, b] \tag{1}$$

with initial conditions:

$$y(a) = y_a, \qquad y'(a) = y'_a \tag{2}$$

where y_a and y'_a are known real values and $f(x)$, $g(x)$, $r(x)$ are functions on I which ensure the existence and uniqueness of solution of problem (1)–(2).

In the fields of applied mathematics and physics (engineering, medical, financial, population dynamics and biological sciences) second-order ordinary initial value problems (IVPs) are often encountered. For example, they occur in the general theory of electrical circuits [1], in the aerodynamic inverse shape design problem [17], and frequently in celestial mechanics [20].

In most cases, this class of equations cannot be solved analytically, thus only approximate solutions can be expected. For this reason a large number of methods for the numerical solution of problem (1)–(2) have been proposed in the literature (see for instance [8,19]).

Some approaches of solving problem (1)–(2) include Nystrom, Runge–Kutta, linear multistep or predictor corrector methods [22,23,28], a combination of Runge–Kutta and multistep procedures [2], deferred-correction methods [7,18]. Recently, many methods for the numerical solution of second order IVPs have been developed by collocation and interpolation technique [3,4,21]. Continuous or piecewise polynomials are very mathematical power tools as they can be differentiated and

* Tel.: +39 0984 496488.
E-mail address: anna.napoli@unical.it.

http://dx.doi.org/10.1016/j.apnum.2015.08.011
0168-9274/

integrated easily and can approximate a great variety of functions to any accuracy desired. For these reasons, for the numerical solution of problem (1)–(2), many methods have been developed which use polynomial basis, such as spline functions [27,33], Bernstein polynomials [5], Chebyshev polynomials [13,14], Hermite polynomials [9,21], Legendre polynomials [43], Taylor polynomials [26,32], Bernoulli polynomials [35].

Bernoulli polynomials have been extensively used for solving IVPs [37], BVPs [24,36], high-order linear and nonlinear Fredholm and Volterra integro-differential equations [6,39], complex differential equations [41] partial differential equations [38,40,42].

In the present paper we also shall use Bernoulli polynomials to derive a spectral method which produces global approximations to the solution $y(x)$ in the form of piecewise polynomial functions. The method combines a recurrence relation for successive derivatives of a solution of (1) with collocation principle and provides an explicit expression of the approximating polynomial of the desired degree $\nu \geq 2$. We will show how, by the use of this kind of polynomials, we can construct a collocation method which is able to produce better results then other existing methods.

The paper is organized as follows: in Section 2 we give a short introduction to Bernoulli polynomials. We also give some important known results, required for our subsequent development, among which is an expansion in Bernoulli polynomials for real functions possessing a sufficient number of derivatives [12]. In Section 3 we present the new method. Section 4 is devoted to the study of stability and error. Finally, in Section 5, as examples, we report some numerical results. The approximate solutions of the considering problems are compared with the solutions obtained by other methods. A reliable good accuracy is obtained in all the considered cases.

2. Preliminaries

The Bernoulli polynomials of degree m, $B_m(x)$, can be defined in various ways over the interval $[0, 1]$. One of them is by the following recursive formulas [12]

$$\begin{cases} B_0(x) = 1 \\ B'_m(x) = mB_{m-1}(x) \qquad m \geq 1 \\ \int_0^1 B_m(x)\,dx = 0 \qquad m \geq 1. \end{cases}$$

The Bernoulli numbers are $B_m = B_m(0)$, $m \geq 0$, and the periodic Bernoulli function of order m, $B^*_m(x)$, are usually defined by the relations

$$\begin{cases} B^*_m(x) = B_m(x) \qquad 0 \leq x < 1 \\ B^*_m(x+1) = B^*_m(x)\,. \end{cases}$$

$B^*_1(x)$ is a discontinuous function with a jump of -1 at each integer. For $m > 1$, $B^*_m(x)$ is a continuous function.

In [12] the following theorem has been proved

Theorem 1. *If $f(x)$ is a real function of class C^ν, $\nu \geq 1$, in the interval $[a, b]$, then, for any $x \in [a, b]$ we have*

$$f(x) = P_\nu(x) + R_\nu(f, x)$$

where

$$P_\nu(x) = f(a) + \sum_{k=1}^{\nu} S_k\left(\frac{x-a}{H}\right)\frac{H^{k-1}}{k!}\left(f^{(k-1)}(b) - f^{(k-1)}(a)\right)$$

$H = b - a$,

$$S_k\left(\frac{x-a}{H}\right) = B_k\left(\frac{x-a}{H}\right) - B_k, \qquad k = 1, \ldots, \nu$$

and

$$R_\nu(f, x) = \frac{H^{\nu-1}}{\nu!}\int_a^b f^{(\nu)}(t)\left[(-1)^\nu B_\nu\left(\frac{t-a}{H}\right) - B^*_\nu\left(\frac{x-t}{H}\right)+\right]dt\,.$$

Remark 1. It can be proved that $P_\nu(a) = f(a)$, $P_\nu(b) = f(b)$.

For the bounds of truncation error $R_\nu(f, x)$ the following result holds [15]: