Applied Numerical Mathematics 90 (2015) 1–8

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Quasi-orthogonality and real zeros of some ${}_2F_2$ and ${}_3F_2$ polynomials

S.J. Johnston [a], K. Jordaan [b,*,1]

[a] Department of Mathematical Sciences, University of South Africa, PO Box 392, UNISA, 0003, South Africa
[b] Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria, 0002, South Africa

ARTICLE INFO

Article history:
Received 2 May 2014
Received in revised form 9 November 2014
Accepted 30 November 2014
Available online 3 December 2014

Keywords:
Hypergeometric polynomials
Quasi-orthogonal polynomials
Zeros
${}_3F_2$ polynomials
${}_2F_2$ polynomials

ABSTRACT

In this paper, we prove the quasi-orthogonality of a family of ${}_2F_2$ polynomials and several classes of ${}_3F_2$ polynomials that do not appear in the Askey scheme for hypergeometric orthogonal polynomials. Our results include, as a special case, two ${}_3F_2$ polynomials considered by Dickinson in 1961. We also discuss the location and interlacing of the real zeros of our polynomials.

1. Introduction

A sequence $\{P_n\}$ of real polynomials of exact degree $n \in \mathbb{N}$ is orthogonal with respect to a positive-definite moment functional $\mathcal{L}$ if (cf. [3])

$$\mathcal{L}\big[R_m(x)R_n(x)\big] = 0 \quad \text{for } m \in \{0, 1, \dots, n-1\}.$$

A well-known consequence of orthogonality is that the n zeros of $P_n(x)$ are real and simple and lie in the supporting set of $\mathcal{L}$ (cf. [3]). The zeros of P_n depart from the supporting set of $\mathcal{L}$ in a specific way when the parameters are changed to values where the polynomials are no longer orthogonal and this phenomenon can be explained in terms of the concept of quasi-orthogonality.

We say that a polynomial sequence $\{R_n\}$ is quasi-orthogonal of order $r \geq 1$, $r \in \mathbb{N}$ with respect to a moment functional $\mathcal{L}$ if

$$\mathcal{L}\big[R_m(x)R_n(x)\big] = 0, \quad |n-m| \geq r+1$$
$$\exists\, s \geq r \text{ such that } \mathcal{L}\big[R_{s-r}(x)R_s(x)\big] \neq 0.$$

It is equivalent to say that

* Corresponding author.
E-mail addresses: johnssj@unisa.ac.za (S.J. Johnston), kjordaan@up.ac.za (K. Jordaan).

1 Research by the second author is partially supported by the National Research Foundation of South Africa.

http://dx.doi.org/10.1016/j.apnum.2014.11.008
0168-9274/

$$\mathcal{L}\big[x^m R_n(x)\big] = 0, \quad m \in \{0, 1, \ldots, n-(r+1)\},\ n \geq r+1$$
$$\exists\, s \geq r \text{ such that } \mathcal{L}\big[x^{s-r} R_s(x)\big] \neq 0.$$

Furthermore, R_n has at least $n-r$ distinct, real zeros in the supporting set of $\mathcal{L}$ (cf. [3]).

Quasi-orthogonal polynomials of order 1 were first introduced by Riesz [22] in 1923 in his solution of the Hamburger moment problem and Fejér [13] considered quasi-orthogonality of order 2 in 1933. In 1937, Shohat [23] generalised the concept of quasi-orthogonality to any order and showed that whenever there exists an orthogonal polynomial sequence $\{P_n\}$ for $\mathcal{L}$, then $\{R_n\}$ being a quasi-orthogonal polynomial sequence of order $r \geq 1$ with respect to $\mathcal{L}$, is equivalent to

$$R_n(x) = \sum_{\nu=n-r}^{n} c_{n,n-\nu} P_\nu(x), \quad n \in \{r, r+1, \ldots\}, \tag{1.1}$$

whilst

$$R_n(x) = \sum_{\nu=0}^{n} c_{n,n-\nu} P_\nu(x), \quad n \in \{0, \ldots, r-1\},$$

and $\exists\, s \geq r$ such that $c_{s,s-r} \neq 0$.

A more general definition of quasi-orthogonality was given in 1957 by Chihara (cf. [2]), who discussed quasi-orthogonality in the context of three-term recurrence relations, proving that a quasi-orthogonal polynomial of any order r satisfies a three-term recurrence relation whose coefficients are polynomials of appropriate degrees. Draux [5] proved the converse of one of Chihara's results and Dickinson [4] improved Chihara's result by deriving a system of recurrence relations that is both necessary and sufficient for quasi-orthogonality. Dickinson applied this method to some special cases of Sister Celine's polynomials

$$f_n(a,x) = {}_3F_2\left(\begin{matrix} -n, n+1, a \\ \frac{1}{2}, 1 \end{matrix}; x\right) = \sum_{m=0}^{n} \frac{(-n)_m (n+1)_m (a)_m}{(\frac{1}{2})_m (1)_m} \frac{x^m}{m!}$$

and proved that $f_n(\frac{3}{2}, x)$ and $f_n(2, x)$ are quasi-orthogonal of order 1 on the interval $(0, 1)$ with respect to the weight functions $(1-x)$ and $x^{-1/2}(1-x)^{3/2}$ respectively. Algebraic properties of the linear functional associated to quasi-orthogonality are given in [5,18–20]. More recent results, particularly on the zeros of order 1 and 2 quasi-orthogonal polynomials, are due to Brezinski, Driver and Redivo-Zaglia [1] and Joulak [16]. For the convenience of the reader, we summarise some of these results.

Lemma 1.1. *Let $\{P_n\}$ be real, monic polynomials of exact degree n that are orthogonal with respect to a positive-definite moment functional $\mathcal{L}$ with supporting set (a, b) and let $x_{i,n}$, $i = 1, 2, \ldots, n$, be the zeros of $P_n(x)$ and y_i, $i = 1, 2, \ldots, n$, the zeros of $R_n(x)$, where*

$$R_n(x) = P_n(x) + a_n P_{n-1}(x)$$

with $a_n \neq 0$. Let $f_n(x) = P_n(x)/P_{n-1}(x)$. Then

(a) $y_1 < a$ if and only if $-a_n < f_n(a) < 0$;
(b) $b < y_n$ if and only if $-a_n > f_n(b) > 0$;
(c) R_n has all its zeros in (a, b) if and only if $f_n(a) < -a_n < f_n(b)$;
(d) $x_{i,n} < y_i < x_{i,n-1}$ for $i = 1, \ldots, n-1$, and $x_{n,n} < y_n$ if and only if $a_n < 0$;
(e) $x_{i-1,n-1} < y_i < x_{i,n}$ for $i = 2, \ldots, n$ and $y_1 < x_{1,n}$ if and only if $a_n > 0$;
(f) $y_{1,n+1} < y_{1,n} < y_{2,n+1} < \cdots < y_{n,n+1} < y_{n,n} < y_{n+1,n+1}$ if and only if $f_{n+1}(y_{n,n}) + a_{n+1} < 0$ when $a_n < 0$ or $f_{n+1}(y_{1,n}) + a_{n+1} > 0$ when $a_n > 0$.

Proof. Parts (a), (b) and (c) are proved in [16, Theorem 4], parts (d) and (e) in [16, Theorem 5] and (f) in [16, Theorem 6]. □

Lemma 1.2. *Let $\{P_n\}$ be real polynomials of exact degree n that are orthogonal with respect to a positive-definite moment functional with supporting set (a, b), and let $x_{i,n}$, $i = 1, 2, \ldots, n$, be the zeros of $P_n(x)$ and y_i, $i = 1, 2, \ldots, n$, the zeros of $R_n(x)$, where*

$$R_n(x) = P_n(x) + a_n P_{n-1}(x) + b_n P_{n-2}(x)$$

with $b_n \neq 0$. Let $f_n(x) = P_n(x)/P_{n-1}(x)$. Then

(a) if $b_n < 0$ then all of the zeros of R_n are real and distinct and at most two of them lie outside the interval (a, b).

In particular,