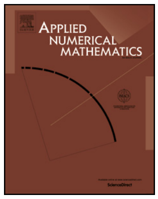
Applied Numerical Mathematics 90 (2015) 55–76

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum



A Gautschi time-stepping approach to optimal control of the wave equation ☆

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ARTICLE INFO

Article history:
Received 12 April 2014
Received in revised form 2 December 2014
Accepted 4 December 2014
Available online 10 December 2014

Keywords:
Gautschi time-stepping
Optimal control
Wave equation
Trigonometric operators
Finite element and spectral methods

ABSTRACT

A Gautschi time-stepping scheme for optimal control of linear second order systems is proposed and analyzed. Convergence rates are proved and shown to be valid in numerical experiments. The temporal discretization is combined with finite element and spectral based spatial discretizations, which are compared among themselves.

1. Introduction

This work is devoted to developing a Gautschi time stepping approach for optimal control problems associated with second order equations, including in particular the wave equation. Solving optimal control problems numerically necessitates to frequently solve the state equation and its adjoint, and hence an efficient method for the latter is indispensable. Compared to optimal control of diffusion systems, the numerical treatment of optimal control of the wave equation has received relatively little attention so far. We refer to [10,9] where the dual weighted residual method for space-time discretization was developed, including as particular case the Crank–Nicolson discretization in time and first order finite element discretization in space. This approach has the desirable property that first discretizing the infinite dimensional optimal control problem and subsequently solving the necessary optimality conditions commutes with first setting up the necessary optimality conditions for the infinite dimensional problem and subsequently discretizing them.

In the present work the focus is put on using a Gautschi scheme for temporal discretization. It will be combined with different spatial discretizations including finite element and spectral techniques. Gautschi integrators have received a considerable amount of attention due to their desirable property that their step sizes are not restricted by the spectral properties of the underlying dynamical system. This is of particular interest for systems which allow highly oscillatory solutions. Gautschi type methods are constructed on the basis that they integrate linear systems with constant inhomogeneities exactly. We refer to [5,4,6,8] and the references given there for further properties of Gautschi techniques. For Gautschi-methods, we

☆ Both authors were supported in part by the Austrian Science Fund (FWF) under grant SFB F32 (SFB "Mathematical Optimization and Applications in Biomedical Sciences").

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http://dx.doi.org/10.1016/j.apnum.2014.12.001
0168-9274/

can also show that discretizing before or after carrying out the optimization step, we obtain the same finite dimensional systems, for the class of spatial discretizations which we shall consider.

The paper is organized as follows. Section 2 contains the problem statement, first order optimality conditions and a brief recollection of cosine operators. The Gautschi time-stepping scheme in an infinite dimensional setting is presented in Section 3. Section 4 contains its analysis for the optimal control problem, with emphasis on the inexact conjugate gradient method for its numerical realization. Numerical results, highlighting convergence rates and comparisons between different spatial discretizations are given in Section 5.

2. Problem formulation and preliminaries

2.1. Problem formulation

Let $V \subset H \subset V'$ be a Gelfand triple of real separable Hilbert spaces and let $T > 0$. Further let $A : V \to V'$ be a V elliptic operator, and consider for vectors $y_0 \in V$, $y_1 \in H$ and $f \in L_2(0,T;H)$ the abstract wave equation

$$\begin{aligned}&\frac{\partial^2 y}{\partial t^2}(t) = Ay(t) + f(t) \quad \text{for } t \in (0,T),\\ &y(0) = y_0,\\ &\frac{\partial y}{\partial t}(0) = y_1.\end{aligned} \tag{W}$$

Definition 2.1 *(Weak solution).* We say that $y \in L_2(0,T;V)$ is a *weak solution* of (W) iff $y_t \in L_2(0,T;H)$, $y_{tt} \in L_2(0,T;V')$,

$$\left\langle \frac{\partial^2 y}{\partial t^2}(t), \varphi \right\rangle_{V',V} = \langle Ay(t), \varphi\rangle_{V',V} + \langle f(t), \varphi\rangle_{V',V} \quad \text{for all } \varphi \in V, \text{ and } t \in (0,T),$$

and $y(0) = y_0$, $y_t(0) = y_1$.

Existence and uniqueness of a weak solution to (W) are well understood (see e.g. [19, Chapter 29, p. 436]). The solution operator $S_W : L_2(0,T;H) \times V \times H \to L_2(0,T;H)$ of the wave equation, which maps (f, y_0, y_1) to the solution y of (W), is continuous (see [19, p. 437]).

For $\beta \in \mathcal{L}(L_2(0,T;H))$, $\tilde{z} \in L^2(0,T;H)$ and $\alpha > 0$ we consider the optimal control problem

$$\begin{aligned}&\min_{y,u \in L_2(0,T;H)} \frac{1}{2}\|y - z\|^2_{L_2(0,T;H)} + \frac{\alpha}{2}\|u\|^2_{L_2(0,T;H)},\\ &\text{s.t. } \frac{\partial^2 y}{\partial t^2} = Ay + \beta u,\\ &y(0,x) = y_0(x), \qquad \frac{\partial y}{\partial t}(0,x) = y_1(x).\end{aligned} \tag{OC}$$

Define the solution operator $S : L_2(0,T;H) \to L_2(0,T;H)$ associated to the wave equation by $Su := S_W(\beta u, y_0, y_1) = y$, with y is solution to (W). We arrive at the reduced problem

$$\min_{u \in L_2(0,T;H)} \frac{1}{2}\|Su - \tilde{z}\|^2_{L_2(0,T;H)} + \frac{\alpha}{2}\|u\|^2_{L_2(0,T;H)}. \tag{1}$$

It is well known that (1) has a unique solution, see e.g. [18, p. 40]. From now on we may assume without loss of generality that $y_0, y_1 = 0$, since we can express y by $y = Su = S_W(\beta u, 0, 0) + S_W(0, y_0, y_1) =: y_I + y_H$. Hence $y - \tilde{z} = y_I - (\tilde{z} - y_H)$. Now we can replace $\tilde{z}$ in the original cost-functional by $z = \tilde{z} - y_H$ and simultaneously replace $S : L^2(0,T;H) \to L^2(0,T;H)$ by $S_W(\beta u, 0, 0)$ arriving at

$$J(u) := \frac{1}{2}\|Su - z\|^2_{L_2(0,T;H)} + \frac{\alpha}{2}\|u\|^2_{L_2(0,T;H)}. \tag{2}$$

The Gateaux derivative $J'(u)$ is given by

$$J'(u) = \big(S^*S + \alpha I\big)u - S^*z,$$

where I is the identity operator. Thus the first order necessary and sufficient optimality condition is given by the operator equation

$$\mathcal{H}u := \big(S^*S + \alpha I\big)u = S^*z. \tag{3}$$

Solving it efficiently will be in the focus of the further considerations.