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Analysis of some projection method based preconditioners for models of incompressible flow

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A B S T R A C T

In this paper, several projection method based preconditioners for various incompressible flow models are studied. In the derivations of these projection method based preconditioners, we use three different types of the approximations of the inverse of the Schur complement, i.e., the exact inverse, the Cahouet–Chabard type approximation and the BFBt type approximation. We illuminate the connections and the distinctions between these projection method based preconditioners and those related preconditioners. For the preconditioners using the Cahouet–Chabard type approximation, we show that the eigenvalues of the preconditioned systems have uniform bounds independent of the parameters and most of them are equal to 1. The analysis is based on a detailed discussion of the commutator difference operator. Moreover, these results demonstrate the stability of the staggered grid discretization and reveal the effects of the boundary treatment. To further illustrate the effectiveness of these projection method based preconditioners, numerical experiments are given to compare their performances with those of the related preconditioners. Generalizations of the projection method based preconditioners to other saddle point problems are also discussed.

1. Introduction

We consider to solve the incompressible Stokes equations

$$\begin{cases} \rho \dfrac{\partial \mathbf{u}}{\partial t} - \nabla \cdot (\mu \nabla \mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega, \\ -\nabla \cdot \mathbf{u} = g & \text{in } \Omega, \end{cases} \tag{1}$$

subject to suitable boundary conditions on $\partial\Omega$. Here, $\mathbf{u}(\mathbf{x}, t)$ is the velocity, $p(\mathbf{x}, t)$ is the pressure, ρ is a constant density function, $\mu(\mathbf{x}, t)$ is the viscosity, $\mathbf{f}$ may include an external force and the nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$, g is 0 or a source term. For simplicity, we assume that $g = 0$ and μ is constant.

In the literature, there has been many methods for solving (1). In the original projection algorithm proposed by Chorin and Temam [7,38], the viscous term is treated explicitly. In some later developed projection methods, e.g. [21,32,33], the viscous term is solved implicitly and there is a correction term in the pressure updating step. The advantages of these projection algorithms exist in that the computations of the velocity and the pressure are decoupled into several Poisson solvers. Compared with these projection methods, many other methods couple the computations of $\mathbf{u}$ and p together.

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However, coupled methods call for effective preconditioners for the resulted saddle point systems. In a recent work [13], the author proposed to use a pressure-correction projection method to be the preconditioner of the saddle point problem resulted from the coupled schemes for $\mathbf{u}$ and p. By taking the advantages of the projection method, it is shown that the preconditioner is effective and efficient [13]. However, we note that the preconditioner in [13] is restricted to the time dependent Stokes case and the corresponding theoretical justification is absent. In this work, we have two aspects of interests. On one aspect, we will further explore the advantages of the projection method. On the other aspect, we will illuminate the connections and the distinctions between these projection method based preconditioners and those related preconditioners [3,2,10,19,23,25–27,31,29]. From our discussion, it will be found that the preconditioner used in [13] is closely related to the Cahouet–Chabard type approximation of the inverse of the Schur complement. In addition, motivated by the related works [10,27,19], we derive and study the other two pressure-correction projection method based preconditioners. Among them, one uses the exact inverse of the Schur complement, the other uses the BFBt type approximations of the inverse of the Schur complement [10]. We adopt a preconditioned GMRES method to solve the resulted saddle point systems. To gain the information of the convergence rate, we analyze the spectrum of the preconditioned system. The effectiveness, the advantages, the disadvantages of each preconditioner will also be discussed. In particular, we highlight the connections and distinctions between the projection method based preconditioners and other preconditioners. It is observed that if the Stokes model degenerates to the mixed form of an elliptic operator, all projection method based preconditioners recover the mixed form of the elliptic operator, no matter which type of boundary conditions is imposed. Furthermore, for the preconditioners using the Cahouet–Chabard approximation, we have the following results: For both the steady and unsteady Stokes problems, it is shown that the preconditioned systems are well conditioned. More precisely, the nontrivial eigenvalues of the preconditioned systems have uniform lower and upper bounds which are independent of the mesh refinement and the physical parameters. Furthermore, the multiplicities of the non-unitary eigenvalues of the preconditioned system are derived based on a detailed analysis of the commutator difference operator. Specifically, for the two dimensional Stokes problems with Dirichlet boundary, if there are n cells along each direction, it is shown that there are at most $4(n-1)$ eigenvalues not equal to 1. If the boundary conditions are periodic, the preconditioned operator is an identity operator. Compared with the existing works for the analysis of the Cahouet–Chabard preconditioner using the finite element discretization, e.g. [25,31,29], our analysis takes a different strategy (cf. Remark 2).

This paper is organized as follows. In Section 2, we present the time and spatial discretization, the boundary treatment, the projection method based preconditioners, the corresponding matrix representations, and the related preconditioners. In Section 3, eigenvalue analysis of the preconditioned systems is presented. In Section 4, numerical experiments are given to compare the performances of different preconditioners. In the last section, we discuss how to generalize the projection method based preconditioners to other saddle point problems or using other types of spatial discretizations.

2. Discretizations and preconditioners

2.1. Time and spatial discretizations

To have better stability and accuracy properties [14,28], an implicit time stepping scheme is adopted in this paper. We apply the following backward Euler scheme.

$$\begin{cases} \dfrac{\rho}{\Delta t}\left(\mathbf{u}^{k+1}-\mathbf{u}^{k}\right) - \nabla\cdot\left(\mu\nabla\mathbf{u}^{k+1}\right) + \nabla p^{k+1} = \mathbf{f}^{k+1}, \\ -\nabla\cdot\mathbf{u}^{k+1} = 0. \end{cases} \tag{2}$$

Here, $\mathbf{u}^k$ and p^k are the approximate solution of $\mathbf{u}$ and p at $t = k\Delta t$.

The staggered grid (i.e. marker-and-cell or MAC [16]) method is applied for the spatial discretization. In this work, we assume that the computational domain Ω is a two dimensional rectangle and there are n_x and n_y cells along the x-direction and the y-direction respectively. For simplicity, let us further assume that $\overline{\Omega} = [0,1]^2$ and $h_x = h_y = h$ (and therefore $n_x = n_y = n$). To present the staggered grid discretization, we introduce the following sets.

$$\overline{\Omega}_1 = \left\{\left(\left(i+\frac{1}{2}\right)h, jh\right) : 0 \le i \le n, 0 \le j \le n+1\right\},$$

$$\overline{\Omega}_2 = \left\{\left(ih, \left(j+\frac{1}{2}\right)h\right) : 0 \le i \le n+1, 0 \le j \le n\right\},$$

$$\Omega_3 = \left\{(ih, jh) : 1 \le i \le n, 1 \le j \le n\right\}.$$

As shown in Fig. 1, $\overline{\Omega}_1$, $\overline{\Omega}_2$ and $\overline{\Omega}_3$ are the point sets for u, v and p respectively. The divergence of $\mathbf{u} = (u, v)^T$ is approximated at cell centers by $D\mathbf{u} = D^x u + D^y v$ with

$$\left(D^x u\right)_{i,j} = \frac{u_{i+1/2,j} - u_{i-1/2,j}}{h}, \qquad \left(D^y v\right)_{i,j} = \frac{v_{i,j+1/2} - v_{i,j-1/2}}{h}.$$