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Numerical evaluation of new quadrature rules using refinable operators

E. Pellegrino

DIIIE, University of L'Aquila, Via Gronchi, 18, 67100 L'Aquila, Italy

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ABSTRACT

This paper concerns the construction of quadrature rules based on the use of suitable refinable quasi-interpolatory operators introduced here. Convergence analysis of the obtained quadrature rules is developed and numerical examples are included.

1. Introduction

It is known that quasi-interpolatory operators play an important role in the approximation of functions, in the numerical solution of integrals or, more in general, of integral equations. Frequently, the quasi-interpolatory operators are based on the use of polynomial or spline bases and provide local tools for practical and effective approximation of functions or discrete data [8,11,13–15].

On the other hand, refinability, which is at the base of subdivision schemes [2] as well as wavelets [3], represents a starting point for the construction of approximation methods. In the last years, such methods have enabled one to obtain a variety of useful applications, ranging from the solution of partial differential equations to the reproduction of curves and surfaces.

The aim of this paper is to present a new quadrature rule, based on a quasi-interpolatory refinable operator approximating smooth functions f with order of compatible accuracy to the best refinable function approximation. The quasi-interpolatory refinable operators is constructed using useful class of refinable functions.

The way to obtain operators with such a property is to require that they reproduce appropriate classes of polynomials. Using such a property a quasi-interpolatory operator has been constructed which allowed us to obtain a quadrature rule with a high degree of accuracy.

A similar operator has been studied in [8,9] and has been applied in [10] to construct a quadrature rule for evaluating singular integrals. It is possible to verify that the quadrature rule constructed in this paper has a degree of accuracy higher than that one obtained with the operator constructed in [8] and the comparison will be done in Section 6.

This paper is organized as follows: in Section 2, some preliminaries on refinable functions are given. In Section 3 useful relations on the coefficients of reproducibility rule are developed. In Section 4 we construct the operator. In Section 5 we develop the quadrature rule associated to the operator and we prove its convergence; finally, in Section 6, some numerical

E-mail address: enza.pellegrino@univaq.it.

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results are shown and comparisons with those which could be obtained using the quasi-interpolatory operator constructed in [8].

2. Preliminaries

A refinable function is the solution of a refinement equation

$$\varphi(x)=\sum_{k\in\mathbb{Z}} b_k\varphi(2x-k), \tag{1}$$

with the mask $\mathbf{b}=\{b_k,\ k\in\mathbb{Z}\}$ and dilation factor 2.

In the following, we shall consider a class of refinable functions that are compactly supported in $I_n:=[0,n+1]$, n a given integer ≥ 2, has a finite *mask* $\mathbf{b}=\{b_k\}_{k=0,\dots,n+1}$ with real coefficients. The corresponding symbol of mask defined as $p(z)=\sum_k b_k z^k$, $z\in\mathbb{C}$, with $p(-1)=0$, $p(1)=2$, is a Hurwitz's polynomial, that is left half plane stable (all its roots have negative real part). It is known that Hurwitz's condition implies $b_k>0$, $k=0,1,\dots,n+1$, and also assures the existence on I_n of a unique continuous solution φ, positive in $(0,n+1)$, vanishing otherwise, [5]. Moreover, the following Schoenberg–Whitney type condition

$$\det_{r,s=1,2,\dots,p}\varphi(t_r-k_s)>0 \quad\Longleftrightarrow\quad k_r<t_r<k_r+n, \tag{2}$$

holds, with $t_r\in\mathbb{R}$, $k_s\in\mathbb{Z}$. In this case φ is called ripplet [2].

The system $\Phi=\{\varphi(x-k),k\in\mathbb{Z}\}$, which is linearly independent, satisfies the following properties

$$\sum_{k\in\mathbb{Z}}\varphi(x-k)=1, \qquad \int_{\mathbb{R}}\varphi(x)dx=1, \tag{3}$$

and the system is totally positive (TP), that is any of its collocation matrix is totally positive.

If for some integer m, $0\leq m\leq n-1$, the symbol can be written as

$$p(z)=(1+z)^{m+1}q(z), \qquad q(1)=2^{-m},$$

the function $\varphi\in C^m(\mathbb{R})$ and then the system Φ reproduces polynomials up to degree m, that is

$$x^s=\sum_{k\in\mathbb{Z}}\xi_k^{(s)}\varphi(x-k), \quad x\in\mathbb{R},\ \xi_k^{(s)}\in\mathbb{R},\ s=0,1,\dots,m. \tag{4}$$

We recall that the coefficients $\xi_k^{(s)}$, in (4), can be obtained by

$$\xi_k^{(s)}=\sum_{r=0}^{s}\binom{s}{r}k^{s-r}C_r, \tag{5}$$

with

$$C_0:=1, \qquad C_r=\sum_{q=1}^{r}(-1)^{q-1}\binom{r}{q}\mu_q C_{r-q}, \tag{6}$$

where $\mu_q:=\int_{\mathbb{R}}x^q\varphi(x)dx$ [7].

Since our interest is toward the evaluation of integrals on finite intervals, we start considering the basis on I_n, said Φ_0, obtained by restricting Φ to the interval I_n. We observe that Φ_0 presents some drawbacks, in particular it introduces some discontinuities at the endpoints, because the edge functions are truncated, more, for the same reason, it gives rise to some numerical instabilities.

Due to the main property of Φ, that is the total positivity, we may obtain a basis on I_n, said W_0, which, simultaneously, is totally positive and reproduces polynomials. In fact, $W_0=\{w_{0k},k=0,1,\dots,2n\}$ whose entries sum up to 1, is the normalized B-basis associated with Φ_0 by the linear transformation

$$\Phi_0=W_0A, \tag{7}$$

where A is a suitable non-singular, totally positive and stochastic matrix [1]. Also, A is the $(2n+1)\times(2n+1)$ matrix of the form

$$A:=\begin{pmatrix} B & & O\\ & 1 & \\ O & & C\end{pmatrix}.$$