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Finite element methods for elliptic optimal control problems with boundary observations

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ABSTRACT

We study in this paper the finite element approximations to elliptic optimal control problems with boundary observations. The main feature of this kind of optimal control problems is that the observations or measurements are the outward normal derivatives of the state variable on the boundary, this reduces the regularity of solutions to the optimal control problems. We propose two kinds of finite element methods: the standard FEM and the mixed FEM, to efficiently approximate the underlying optimal control problems. For both cases we derive a priori error estimates for problems posed on polygonal domains. Some numerical experiments are carried out at the end of the paper to support our theoretical findings.

1. Introduction

In this paper, we consider the following elliptic optimal control problems with boundary observations:

$$\min_{u\in U_{ad}} J(y,u)=\frac{1}{2}\int_{\Gamma}(\partial_{n_A}y-z_d)^2ds+\frac{\alpha}{2}\int_{\Omega_U}u^2dx \tag{1.1}$$

subject to

$$\begin{cases}-\mathrm{div}(A\nabla y)=f+Bu & \text{in } \Omega,\\ y=0 & \text{on } \Gamma,\end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^2$ is an open bounded, convex domain with boundary $\Gamma=\partial\Omega$, $z_d\in L^2(\Gamma)$ and $f\in L^2(\Omega)$ are given functions which can be sufficiently smooth if needed, $\alpha>0$ is a regularization parameter, A is the coefficient matrix whose property will be stated later, $\partial_{n_A}y$ is defined as $A\cdot\nabla y\cdot\boldsymbol{n}$ in which $\boldsymbol{n}$ is the outward normal vector on the boundary, $\Omega_U\subset\Omega$ is the sub-domain where the control acts, and B is the control operator defined as $B:L^2(\Omega_U)\to L^2(\Omega)$ which usually takes the form χ_{Ω_U} with χ_{Ω_U} the characteristic function of the sub-domain $\Omega_U\subset\Omega$. We denote the set of admissible controls by

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$$U_{ad} = \left\{ u \in L^2(\Omega_U) \colon a \leqslant u \leqslant b \text{ a.e. in } \Omega_U \right\},$$

where a and b are real numbers.

There is a kind of inverse problems that aims to convert observed measurements into information about physical object or system, and the applications can be found in many branches, such as geophysics, medical imaging, astronomy and non-destructive testing (see [3,29] for example). The problem studied in this paper comes from a specific inversion analysis in geo-technical engineering: the model is built to derive the body force when we can only measure the stress $\partial_{n_A} y$ on the surface by using Jacking methods or surface relief methods (see, e.g., [1,25]). Here, we simplify the geo-technical system into an elliptic equation since they have similar properties in mathematical analysis.

Meanwhile, this problem can also be viewed as an elliptic optimal control problem with distributed control and observations on the boundary. We refer to [20,21,30] for an overview of optimal control problems. There have been extensive theoretical and numerical studies for finite element approximations of various optimal control problems. For instance, the error analysis for optimal control problems with distributed and boundary controls governed by linear elliptic equations has been established in [14,16]. A variational discretization concept was proposed by Hinze in [19]. A super-convergence property of elliptic optimal control problems was exploited in [26] by a post-processing technique. A posteriori error estimates of residual type were also derived in, e.g., [22,23]. Some recent progress in this area has been summarized in [20,24].

As far as we know, there are only few papers concerning error analysis of elliptic optimal control problems with observations other than distributed ones. Here we should mention the work [11] which considered the finite element approximations of elliptic optimal control problems with point-wise observations and [27] which studied a parameter identification problems with point-wise measurements. In [5] the authors considered the boundary concentrated finite element method for Neumann boundary controls of elliptic equations with Dirichlet boundary observations. In this paper we intend to consider finite element approximations to elliptic optimal control problems with boundary observations. The main feature of this kind of optimal control problems is that the observations are the outward normal derivatives of the state variable on the boundary, this reduces the regularity of solutions to the optimal control problems.

From the maximum principle of the control problems it is easy to find the resemblance between the boundary observation problems and Dirichlet boundary control problems (see e.g., [9,13,17]). For Dirichlet boundary control problems the control acts as the Dirichlet boundary condition of the state equation. Thus, we need to deal with elliptic equations with inhomogeneous Dirichlet data belonging to only $L^2(\Gamma)$, whose weak solution can only be understood in a very weak sense (see [4]). For the finite element approximation of Dirichlet boundary control problems one need to combine standard finite element method with boundary L^2-projection to approximate the Dirichlet data. While for the boundary observation problems, the major difficulty comes from the fact that the adjoint state equation is an elliptic equation with inhomogeneous Dirichlet boundary condition whose value is the partial derivative of the state. Therefore, for the numerical approximation we can borrow some techniques of dealing with Dirichlet boundary control problems to analyze our problem.

In [9] Casas and Raymond considered semi-linear Dirichlet boundary control problems posed on polygonal domains and derived optimal a priori error estimate. Deckelnick, Günther and Hinze have studied the above problem posed on smooth domains in [13] and obtained some super-convergence results. The above mentioned papers are all based on the standard finite element method combining with L^2-projection to deal with the inhomogeneous Dirichlet boundary condition. An alternative method based on mixed variational form was proposed in [17] to approximate the Dirichlet boundary control problems. The advantage of mixed variational form is that the essential boundary condition appears to be a natural one and it is easier for theoretical analysis and numerical realization.

To efficiently approximate the problems (1.1)–(1.2) we use two kinds of finite element methods mentioned above: the standard FEM and the mixed FEM, to approximate the state variable, while the control is discretized by variational discretization concept proposed in [19]. We derive a priori error estimates for both cases. The main result of this paper is as follows:

$$\|u - u_h\|_{0,\Omega_U} \leq Ch, \tag{1.3}$$

where u and u_h are the continuous and discrete optimal controls by using the standard finite element or the mixed finite element approximations, where h is the mesh size of the triangulation. We remark that the above results hold under the additional stronger assumptions that $\Omega = \Omega_U$ and $f \in H^1(\Omega)$ in the case of the mixed finite element approximation. We can observe from the numerical experiments that the standard FEM outperforms the mixed FEM both in computational complexity and accuracy, especially for problems with smooth solutions. However, the mixed FEM provides an option to solve optimal control problems with inhomogeneous Dirichlet boundary conditions or with gradient information in the objective functional. Moreover, for possible extension to practical problems with governing equations of the form of linear elasticity systems, the mixed FEM is more robust compared to the standard FEM when the compliance tensor becomes singular (see [2]).

The novelty of our paper compared to the previous ones is twofold. First, the control problem discussed in this paper has special structure of the objective functional which consists of an observation term for the outward normal derivative of the state on the boundary. This is the first paper which deals with such problem as far as we know. Second, the low regularity of the problem brings some difficulties to the error estimation, especially when the state and adjoint state are both approximated by piecewise linear continuous functions. We remark that although we borrow some techniques from