Applied Numerical Mathematics 95 (2015) 2–14

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Finite element potentials

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ARTICLE INFO

Article history:
Available online 8 July 2014

Keywords:
Grad operator
Curl operator
Divergence operator
Finite elements
First homology group
Graph
Spanning tree

ABSTRACT

We present an explicit and efficient way for constructing finite elements with assigned gradient, or curl, or divergence. Some simple notions of homology theory and graph theory applied to the finite element mesh are basic tools for devising the solution algorithms.

1. Introduction

Determining the necessary and sufficient conditions for assuring that a vector field defined in a bounded and sufficiently smooth three-dimensional domain Ω is the gradient of a scalar potential or the curl of a vector potential is one of the most classical problems of vector analysis.

The answer is well-known, and shows an interesting interplay of differential calculus and topology (see, e.g., Cantarella et al. [3]):

- a vector field is the gradient of a scalar potential if and only if it is curl-free and its line integral is vanishing on all the closed curves that give a basis of the first homology group of $\overline{\Omega}$;
- a vector field is the curl of a vector potential if and only if it is divergence-free and its flux is vanishing across (all but one) the connected components of $\partial\Omega$.

Less interesting is the problem of finding a vector field with assigned divergence f: this problem is very simply solved by taking the gradient of the solution φ of the elliptic problem $\Delta\varphi = f$ in Ω, φ vanishing on the boundary $\partial\Omega$; no compatibility conditions on f are needed, no topological properties of Ω come into play.

However, a less clarified situation takes shape when, given a suitable *finite element* vector field, we want to furnish an explicit and efficient procedure for constructing its *finite element* scalar potential and vector potential. Note also that at this level the construction of a finite element vector field with an assigned divergence comes back on the table: in fact, the gradient of a (standard) finite element approximate solution of $\Delta\varphi = f$ has a distributional divergence which is not a function, and therefore this divergence cannot be equal to an assigned finite element.

The aim of this paper is to furnish a simple and efficient way for constructing finite elements with assigned gradient, or curl, or divergence. Clearly, in numerical computations this is important any time one has to reduce a given problem to an associated one with vanishing data.

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http://dx.doi.org/10.1016/j.apnum.2014.05.014

It is worth noting that the computational cost of all the algorithms we propose depends linearly on the number of unknowns.

2. Notation and preliminary results

Let Ω be a bounded polyhedral domain of $\mathbb{R}^3$ with Lipschitz boundary and let $(\partial\Omega)_0, \dots, (\partial\Omega)_p$ be the connected components of $\partial\Omega$. Consider a tetrahedral triangulation $\mathcal{T}_h = (V, E, F, T)$ of $\overline{\Omega}$. Here V is the set of vertices, E the set of edges, F the set of faces and T the set of tetrahedra in $\mathcal{T}_h$.

We consider the following spaces of finite elements (for a complete presentation, see Monk [7]). The space L_h of continuous piecewise-linear finite elements; its dimension is n_v, the number of vertices in $\mathcal{T}_h$. The space N_h of Nédélec edge elements of degree 1; its dimension is n_e, the number of edges in $\mathcal{T}_h$. The space RT_h of Raviart–Thomas finite elements of degree 1; its dimension is n_f, the number of faces in $\mathcal{T}_h$. The space PC_h of piecewise-constant elements; its dimension is n_t, the number of tetrahedra in $\mathcal{T}_h$.

It is well-known that $L_h \subset H^1(\Omega)$, $N_h \subset H(\text{curl}; \Omega)$, $RT_h \subset H(\text{div}; \Omega)$ and $PC_h \subset L^2(\Omega)$, where

$$H^1(\Omega) = \left\{\phi \in L^2(\Omega) \;\middle|\; \text{grad}\,\phi \in \left(L^2(\Omega)\right)^3\right\},$$
$$H(\text{curl}; \Omega) = \left\{\mathbf{v} \in \left(L^2(\Omega)\right)^3 \;\middle|\; \text{curl}\,\mathbf{v} \in \left(L^2(\Omega)\right)^3\right\},$$
$$H(\text{div}; \Omega) = \left\{\mathbf{v} \in \left(L^2(\Omega)\right)^3 \;\middle|\; \text{div}\,\mathbf{v} \in L^2(\Omega)\right\}.$$

Moreover $\text{grad}\, L_h \subset N_h$, $\text{curl}\, N_h \subset RT_h$ and $\text{div}\, RT_h \subset PC_h$.

Fix a total ordering $v_1, \dots, v_{n_v}$ of the elements of V. This induces an orientation on the elements of E, F and T: if the end points of e_j are v_a and v_b for some $a, b \in \{1, \dots, n_v\}$ with $a < b$, then the oriented edge e_j is denoted by $[v_a, v_b]$, and therefore the unit tangent vector of e_j is given by $\boldsymbol{\tau} = \frac{v_b - v_a}{|v_b - v_a|}$. On the other hand, if the face f has vertices v_a, v_b and v_c with $a < b < c$, the oriented face f is denoted by $[v_a, v_b, v_c]$ and its unit normal vector $\boldsymbol{\nu}$ is obtained by the right hand rule. Finally, if the tetrahedron t has vertices v_a, v_b, v_c and v_d with $a < b < c < d$, the oriented tetrahedron f is denoted by $[v_a, v_b, v_c, v_d]$.

Let us consider a basis of L_h, $\{\Phi_{h,1}, \dots, \Phi_{h,n_v}\}$, such that

$$\Phi_{h,i}(v_j) = \delta_{i,j}$$

for $1 \le i, j \le n_v$, a basis of N_h, $\{\mathbf{w}_{h,1}, \dots \mathbf{w}_{h,n_e}\}$, such that

$$\int_{e_j} \mathbf{w}_{h,i} \cdot \boldsymbol{\tau} = \delta_{i,j}$$

for $1 \le i, j \le n_e$, a basis of RT_h, $\{\mathbf{r}_{h,1}, \dots \mathbf{r}_{h,n_f}\}$, such that

$$\int_{f_j} \mathbf{r}_{h,i} \cdot \boldsymbol{\nu} = \delta_{i,j}$$

for $1 \le i, j \le n_f$, and the basis of PC_h, $\{g_{h,1}, \dots g_{h,n_t}\}$, given by the characteristic functions of the tetrahedron t_i.

In the following we introduce some notions of homology theory (see, e.g., Munkres [8]). We start from the mesh $\mathcal{T}_h = (V, E, F, T)$ on $\overline{\Omega}$, having assigned the orientation to the edges and faces as explained before. The basic concept is that of chain: a 2-chain is a formal linear combination of oriented faces, a 1-chain is a formal linear combination of oriented edges, and a 0-chain is a formal linear combination of vertices, in all cases taking the coefficients in $\mathbb{Z}$. We denote by $C_k(\mathcal{T}_h, \mathbb{Z})$ the set of all the k-chains in $\mathcal{T}_h$, $k = 0, 1, 2$.

Now we can define the boundary operator $\partial_k : C_k(\mathcal{T}_h, \mathbb{Z}) \to C_{k-1}(\mathcal{T}_h, \mathbb{Z})$ for $k = 1, 2$. For the oriented face $f = [v_{a_0}, v_{a_1}, v_{a_2}]$ we have

$$\partial_2 f := [v_{a_1}, v_{a_2}] - [v_{a_0}, v_{a_2}] + [v_{a_0}, v_{a_1}].$$

Analogously for the oriented edge $e = [v_a, v_b]$ we have

$$\partial_1 e := v_b - v_a.$$

We extend the definition of the boundary operator to chains by linearity.

A 1-chain c of $\mathcal{T}_h$ is a 1-cycle if $\partial_1 c = 0$, and is a 1-boundary if there exists a 2-chain C such that $\partial_2 C = c$. Notice that all 1-boundaries are 1-cycles but, in general, not all 1-cycles are 1-boundaries.

Let us denote by $Z_1(\mathcal{T}_h, \mathbb{Z})$ the set of 1-cycles, $Z_1(\mathcal{T}_h, \mathbb{Z}) := \ker(\partial_1)$, and $B_1(\mathcal{T}_h, \mathbb{Z})$ the set of 1-boundaries, $B_1(\mathcal{T}_h, \mathbb{Z}) := \text{im}(\partial_2)$. Two 1-cycles c and c' are called homologous in $\mathcal{T}_h$ if $c - c'$ is a 1-boundary in $\mathcal{T}_h$. If c is homologous to the trivial 1-cycle (namely, it is a 1-boundary), then we say that c bounds in $\mathcal{T}_h$.