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Efficient and accurate implementation of *hp*-BEM for the Laplace operator in 2D

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ABSTRACT

We discuss the accurate and efficient implementation of *hp*-BEM for the Laplace operator in two dimensions. Using Legendre polynomials and their antiderivatives as local bases for the discrete ansatz spaces, we are able to reduce both the evaluation of potentials and the computation of Galerkin entries to the evaluation of basic integrals. For the computation of these integrals we derive recurrence relations and discuss their accurate evaluation. Our implementation of *p*- and *hp*-BEM produces accurate results even for large polynomial degrees ($p > 1000$) while still being efficient. While this work only treats Symm's integral equation for the Laplace operator in 2D, our approach can be used to solve Symm's, hypersingular and mixed integral equations for Laplace, Lamé and Stokes problems in two dimensions.

1. Introduction

The boundary element method (BEM) has proven to be a useful method for the solution of different types of partial differential equations arising in technical applications such as potential theory, linear elasticity or electromagnetism. One main advantage of BEM as compared to other discretization methods for PDEs such as finite element and finite volume methods is the fact that only the boundary of the domain under consideration has to be discretized. Furthermore, BEM can handle unbounded domains in a natural way. High order and *hp*-boundary element methods are of special interest when solutions need to be computed with high accuracy. As compared to lowest order BEM, *p*- and *hp*-versions of BEM lead to higher rates of convergence with respect to the number of degrees of freedom. In many cases, even exponential convergence can be obtained.

However, the implementation of boundary element methods is not straight forward. The main challenge is the numerical evaluation of singular integrals. Several approaches using regularizing transformations and adapted quadrature rules have been proposed in e.g. [20,18,19], or [21]. Furthermore, approaches where the integrals are evaluated analytically for monomial basis functions are discussed in e.g. [14]. For lowest order BEM in two and three dimensions, these approaches work well and have been used for the efficient implementation. Several software packages that implement lowest order BEM in a variety of programming languages reaching from C++ to MATLAB have been published, for example BETL [12], HILBERT [1] or BEM++ [22] to name a few but not all. However, for the implementation of high order and *hp*-BEM, quadrature formulas and analytic formulas for monomial basis functions lead to unsatisfying results. Using monomials as element basis functions leads to Galerkin matrices whose condition numbers grow exponentially in the polynomial degree of the ansatz space. Therefore, in this approach the polynomial degree of the ansatz space is limited. Furthermore, in *p*-BEM polynomials of high orders combined with a singular kernel have to be integrated. This can only be done accurately with high order

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quadrature formulas and for low polynomial degrees. Overall, in previous publications mainly results for polynomial degrees less than 50 have been presented.

In this work we present a new approach for the accurate and efficient evaluation of potentials and Galerkin entries arising in high order and *hp*-BEM in two dimensions. We show that all arising single and double integrals can be transformed to a complex reference situation and derive simple recurrence relations for the evaluation of the resulting integrals. We also discuss how these recurrence relations can be evaluated accurately up to high orders.

Overall our implementation produces accurate results even for polynomial degrees $p > 5000$ and allows for error reduction to the maximum accuracy that can be achieved with double precision computations.

Here, we only discuss BEM for Symm's integral equation, resp. the Laplace operator. However, we emphasize, that our approach can be applied to Lamé [2] and Stokes problems.

2. Model problem and ansatz spaces

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary. Let $\Gamma \subseteq \partial\Omega$ be a relatively open subset of the boundary. We consider Symm's integral equation

$$\mathcal{V}\phi = f \quad \text{on } \Gamma, \tag{1}$$

where the right hand side $f \in H^{1/2}(\Gamma)$ is given and the single layer operator $\mathcal{V}$ is defined by

$$(\mathcal{V}\phi)(x) = -\frac{1}{2\pi}\int_\Gamma \phi(y)\log|x-y|\,ds_y. \tag{2}$$

If the capacity of Ω, $\operatorname{cap}(\Omega)$, is smaller than one, $\mathcal{V}: \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is an elliptic isomorphism and provides a scalar product on $\widetilde{H}^{-1/2}(\Gamma)$ given by $\langle\!\langle \phi, \psi \rangle\!\rangle = \langle \mathcal{V}\phi, \psi\rangle$, where $\langle\cdot,\cdot\rangle$ is the extended L^2 scalar product. Furthermore, the induced norm $|||\cdot|||$ is an equivalent norm on $\widetilde{H}^{-1/2}(\Gamma)$. Finally, the Lax–Milgram theorem ensures the unique solvability of (1) and of the corresponding variational formulation

$$\langle \mathcal{V}\phi, \psi\rangle = \langle f, \psi\rangle \quad \text{for all } \psi \in \widetilde{H}^{-1/2}(\Gamma). \tag{3}$$

As discrete ansatz space for our Galerkin BEM we use the space of piecewise polynomials with respect to a given boundary mesh $\mathcal{T}_h$. Here, $\mathcal{T}_h := \{T_1, \dots, T_{N_E}\}$ is a set of affine closed line segments $T_i \subset \Gamma$ with $|T_i \cap T_j| = 0$ for $i \neq j$ and $\bigcup_{i=1}^{N_E} T_i = \Gamma$. Furthermore, since Ω is a polygonal domain and $\Gamma \subseteq \partial\Omega$, we require that the vertices of $\partial\Omega$ that belong to Γ are nodes of the boundary mesh. For a given boundary mesh $\mathcal{T}_h$ and a polynomial degree vector $p = (p_i)_{i=1\dots,N_E} \in \mathbb{N}_0^{N_E}$, we define the discrete ansatz space by

$$S(\mathcal{T}_h, p) := \big\{\phi \in L^2(\Gamma)\colon\ \phi|_{T_i} \in \mathcal{P}^{p_i}(T_i),\ i = 1, \dots, N_E\big\}, \tag{4}$$

where $\mathcal{P}^n(T_i)$ is the space of all polynomials up to degree n with respect to the arc length on the element T_i. The local mesh width is denoted by $|T_i| = h_i$ for all $T_i \in \mathcal{T}_h$.

As local basis functions for the space $S(\mathcal{T}_h, p)$ we choose Legendre polynomials, which can be defined for $z \in \mathbb{C}$ by

$$\begin{aligned} &P_0(z) = 1, \qquad P_1(z) = z \\ &(k+1)P_{k+1}(z) = (2k+1)zP_k(z) - kP_{k-1}(z), \quad k \geqslant 1. \end{aligned} \tag{5}$$

For each element $T_i = \operatorname{conv}\{A_i, B_i\}$, $A_i, B_i \in \Gamma$, we define the affine mapping $\gamma_i : \widehat{T} = [-1,1] \to T_i$, $\gamma_i(t) = m_i + u_i t$, which parametrizes the element $T_i \in \mathcal{T}$. Here $m_i = (A_i + B_i)/2$ is the midpoint of T_i and $u_i = (B_i - A_i)/2$. Setting $P_k^{(i)} := P_k \circ \gamma_i^{-1}$ we can write the local basis of $S(\mathcal{T}_h, p)$ on T_i as $\{P_0^{(i)}, \dots, P_{p_i}^{(i)}\}$ and overall the basis of $S(\mathcal{T}_h, p)$ can be written as

$$\bigcup_{i=1}^{N_E} \big\{P_0^{(i)}, \dots, P_{p_i}^{(i)}\big\}. \tag{6}$$

The choice of an appropriate basis for the space $S(\mathcal{T}_h, p)$ is critical for the accurate implementation of *p*- and *hp*-BEM with high polynomial degrees. In particular, the condition number of the Galerkin matrix depends on the choice of the basis functions. The condition number of the single layer Galerkin matrix is proportional to p^α with $\alpha \leqslant 2$ if scaled Legendre polynomials are used as element basis functions [10]. Numerical experiments with different choices of basis functions are discussed in [3]. It is shown that, whereas for monomials as local basis functions the condition number grows exponentially, for orthogonal polynomials the condition number grows only algebraically with respect to the polynomial degree p of the ansatz space. If the Galerkin matrix is scaled by its diagonal, we obtain linear growth of the condition number with respect to p for the case of Legendre polynomials as local basis functions.