



Pure Lagrangian and semi-Lagrangian finite element methods for the numerical solution of Navier–Stokes equations

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ABSTRACT

In this paper we propose a unified formulation to introduce Lagrangian and semi-Lagrangian velocity and displacement methods for solving the Navier–Stokes equations. This formulation allows us to state classical and new numerical methods. Several examples are given. We combine them with finite element methods for spatial discretization. In particular, we propose two new second-order characteristics methods in terms of the displacement, one semi-Lagrangian and the other one pure Lagrangian. The pure Lagrangian displacement methods are useful for solving free surface problems and fluid–structure interaction problems because the computational domain is independent of the time and fluid–solid coupling at the interphase is straightforward. However, for moderate to high-Reynolds number flows, they can lead to high distortion in the mesh elements. When this happens it is necessary to remesh and reinitialize the transformation to the identity. In order to assess the performance of the obtained numerical methods, we solve different problems in two space dimensions. In particular, numerical results for a sloshing problem in a rectangular tank and the flow in a driven cavity are presented.

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1. Introduction

The main goal of the present paper is to introduce new second-order pure Lagrangian and semi-Lagrangian methods for the numerical solution of Navier–Stokes equations. In the scalar case, methods of characteristics for time discretization of convection–diffusion problems are extensively used (see the review paper [17]). These methods are based on time discretization of the material time derivative and were introduced in the beginning of the eighties of the last century combined with finite differences or finite elements for space discretization (see [14,25]). When combined with finite elements they are also called Lagrange–Galerkin methods. In particular, when the characteristics methods are formulated in a fixed reference domain (respectively, in the current domain) they are called pure Lagrangian methods (respectively, semi-Lagrangian methods). In particular, the classical method of characteristics, as introduced in [14] and [25], is semi-Lagrangian and first order in time. There exists an extensive literature studying this characteristics method combined with finite elements applied to scalar convection–diffusion equations. If Δt denotes the time step, h the mesh-size and k the degree of the finite element space, estimates of the form $O(h^k) + O(\Delta t)$ in the $l^\infty(L^2(\mathbb{R}^d))$ -norm are shown in [29] (d denotes the dimension of the spatial domain). Moreover, in [25] error estimates of the form $O(h^k) + O(\Delta t) + O(h^{k+1}/\Delta t)$ in the $l^\infty(L^2(\Omega))$ -norm are obtained under the assumption that the normal velocity vanishes on the boundary of Ω . All of these estimates involve

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constants depending on solution norms. For linear finite elements and for a velocity field vanishing on the boundary, convergence of order $O(h^2) + O(\min(h, h^2/\Delta t)) + O(\Delta t)$ in the $l^\infty(L^2(\Omega))$ -norm is stated in [1], where the constants only depend on the data. In principle, the method of characteristics has been introduced for evolution equations but an adaption to solve stationary convection–diffusion problems has been proposed in [7].

In order to increase the order of time and space approximations, higher order schemes for the discretization of the material derivative and higher order finite element spaces should be used. In [27], a second-order characteristics method for solving constant coefficient convection–diffusion equations with Dirichlet boundary conditions is studied. The Crank–Nicholson discretization has been used to approximate the formulation involving the material time derivative. For a divergence-free velocity field vanishing on the boundary and a smooth enough solution, stability and $O(\Delta t^2) + O(h^k)$ error estimates in the $l^\infty(L^2(\Omega))$ -norm are stated (see also [8] and [9] for further analysis).

Recently, for scalar linear convection–diffusion problems, we have introduced so-called pure Lagrangian methods combined with finite elements. In particular, in [4] and [5] $l^\infty(H^1(\Omega))$ stability and $l^\infty(H^1(\Omega))$ error estimates of order $O(\Delta t^2) + O(h^k)$ were proved for a second-order pure Lagrange–Galerkin method. Moreover, in [13], semi-Lagrangian and pure Lagrangian methods are proposed and analyzed for convection–diffusion equations. Error estimates for Galerkin discretization of a pure Lagrangian formulation and for a discontinuous Galerkin discretization of a semi-Lagrangian formulation are obtained. The estimates are written in terms of the projections constructed in [11] and [12]. In [4] and [5] a pure Lagrangian formulation has been used for more general problems. Specifically, we have considered a (possibly degenerate) variable coefficient diffusive term instead of the simpler Laplacian, general mixed Dirichlet–Robin boundary conditions, and a time dependent domain. Moreover, we have analyzed a scheme with approximate characteristic curves and presented numerical results for pure Lagrangian and semi-Lagrangian methods. In [2] a unified formulation to introduce Lagrangian and semi-Lagrangian methods for solving scalar linear convection–diffusion problems has been proposed and new stability estimates for the pure Lagrangian method proposed in [4] and [5] have been obtained. More precisely, an $l^\infty(H^1)$ -stability estimate independent of the diffusion coefficient has been proved. Besides, if the given velocity field is incompressible, a stability inequality independent of the final time has been shown.

Usually, the unconditional stability of characteristics methods is only proved under the assumption that the inner products in the Galerkin formulation are exactly calculated. This is rarely possible so in practice they have to be calculated by using numerical quadrature. In general, this adds some terms to the final error estimates and, in some cases, it produces the loss of unconditional stability. There are several papers in the literature analyzing the effect of numerical integration in Lagrange–Galerkin methods (see [3,9,19,24,26,29,30]).

In this paper, we introduce a unified formulation to state pure Lagrangian and semi-Lagrangian methods for solving vector nonlinear convection–diffusion equations. More precisely, we are interested in solving the Navier–Stokes equations. For this purpose, we use the mathematical formalism of continuum mechanics (see for instance [22]) following the ideas given in [2].

The paper is organized as follows. In Section 2 the initial-boundary value problem to be solved is posed in a time dependent bounded domain and some hypotheses and notations concerning motions are stated. In Section 3, we introduce a quite general change of variable obtaining a new strong formulation of the problem. Moreover, the standard associated weak problem is obtained. In Section 4, semi-Lagrangian schemes in terms of the velocity are proposed. All these methods arise from the formulation obtained in the previous section. By using this formulation, in Section 5, two new Lagrange–Galerkin schemes in terms of the displacement are proposed, one pure Lagrangian and another one semi-Lagrangian. Finally, in Section 6 numerical examples are presented.

2. Statement of the nonlinear convection diffusion problem. General assumptions and notations

Let Ω be a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) with Lipschitz boundary Γ divided into two parts: $\Gamma = \Gamma^D \cup \Gamma^N$, with $\Gamma^D \cap \Gamma^N = \emptyset$. Let t_0 and T be two non-negative constants and $X : \overline{\Omega} \times [t_0, T] \rightarrow \mathbb{R}^d$ be a *motion* in the sense of Gurtin [22]. In particular, $X \in C^3(\overline{\Omega} \times [t_0, T])$ and for each fixed $t \in [t_0, T]$, $X(\cdot, t)$ is a one-to-one function satisfying

$$\det F(p, t) > 0 \quad \forall p \in \overline{\Omega}, \quad (1)$$

being $F(\cdot, t)$ the *deformation gradient* of $X(\cdot, t)$. We call $\Omega_t = X(\Omega, t)$, $\Gamma_t = X(\Gamma, t)$, $\Gamma_t^D = X(\Gamma^D, t)$ and $\Gamma_t^N = X(\Gamma^N, t)$, for $t \in [t_0, T]$. We assume that $\Omega_{t_0} = \Omega$. Let us introduce the *trajectory* of the motion

$$\mathcal{T} := \{(x, t) : x \in \overline{\Omega}_t, t \in [t_0, T]\}.$$

For each t , $X(\cdot, t)$ is a one-to-one mapping from $\overline{\Omega}$ onto $\overline{\Omega}_t$; hence it has an inverse

$$P(\cdot, t) : \overline{\Omega}_t \rightarrow \overline{\Omega}, \quad (2)$$

such that

$$X(P(x, t), t) = x, \quad P(X(p, t), t) = p \quad \forall (x, t) \in \mathcal{T} \quad \forall (p, t) \in \overline{\Omega} \times [t_0, T]. \quad (3)$$

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