Applied Numerical Mathematics 95 (2015) 99–117

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

# Post-processing procedures for an elliptic distributed optimal control problem with pointwise state constraints ☆

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ARTICLE INFO

*Article history:*
Available online 14 March 2015

*Keywords:*
Elliptic optimal control
State constraints
Quadratic $C^0$ interior penalty method
Fourth order variational inequality
Post-processing

ABSTRACT

We consider an elliptic distributed optimal control problem with state constraints and compare three post-processing procedures that compute approximations of the optimal control from the approximation of the optimal state obtained by a quadratic $C^0$ interior penalty method.



## 1. Introduction

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded convex polygonal domain, $y_d \in L_2(\Omega)$ and $\beta$ be a positive number. We consider a model elliptic distributed optimal control problem with pointwise state constraints (cf. [15]):

$$\begin{aligned} &\text{minimize} \quad J(y,u) = \frac{1}{2}\int_\Omega (y-y_d)^2\,dx + \frac{\beta}{2}\int_\Omega u^2\,dx \\ &\text{over} \quad (y,u) \in H_0^1(\Omega) \times L_2(\Omega) \\ &\text{subject to} \quad \begin{cases} -\Delta y = u & \text{in } \Omega \\ \quad y \le \psi & \text{a.e. in } \Omega \end{cases} \end{aligned} \tag{1.1}$$

We assume that $\psi \in C^2(\bar{\Omega})$ and $\psi > 0$ on $\partial\Omega$. Here and below we will follow standard notation for differential operators and Sobolev spaces that can be found for example in [17,9].

Since $\Omega$ is convex, it follows from the elliptic regularity theory [25,18,35] that the state $y$ belongs to $H^2(\Omega) \cap H_0^1(\Omega)$ and hence, by replacing the control $u$ with $-\Delta y$ in (1.1), we can instead look for the minimizer of the reduced functional

$$G(y) = \frac{1}{2}\int_\Omega (y-y_d)^2\,dx + \frac{\beta}{2}\int_\Omega (\Delta y)^2\,dx$$

in the closed convex set $K = \{y \in H^2(\Omega) \cap H_0^1(\Omega) : y \le \psi \text{ in } \Omega\}$.

☆ This work was supported in part by the National Science Foundation under Grant Nos. DMS-10-16332 and DMS-13-19172.
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http://dx.doi.org/10.1016/j.apnum.2015.03.001
0168-9274/

Note that [26, Theorem 2.2.1]

$$\int_\Omega (\Delta v)(\Delta w)\,dx = \int_\Omega (D^2 v : D^2 w)\,dx \qquad \forall\, v, w \in H^2(\Omega)\cap H^1_0(\Omega),$$

where $D^2 v : D^2 w = \sum_{i,j=1}^2 v_{x_i x_j} w_{x_i x_j}$ is the inner product of the Hessian matrices of $v$ and $w$. Therefore, after a simple manipulation, we have the following equivalent minimization problem:

$$\text{Find}\quad \bar{y} = \operatorname*{argmin}_{y\in K}\left[\frac{1}{2}\mathcal{A}(y, y) - (y_d, y)\right], \tag{1.2}$$

where $(\cdot,\cdot)$ is the inner product of $L_2(\Omega)$ and

$$\mathcal{A}(v, w) = \int_\Omega [\beta(D^2 v : D^2 w) + vw]\,dx. \tag{1.3}$$

Since the symmetric bilinear form $\mathcal{A}(\cdot,\cdot)$ is coercive on $H^2(\Omega)\cap H^1_0(\Omega)$ by a Poincaré–Friderichs inequality [36] and $K$ is a closed convex subset of $H^2(\Omega)\cap H^1_0(\Omega)$ by the Sobolev inequality [1], it follows from the standard theory [32,29,38,23] that (1.2) has a unique solution characterized by the fourth order variational inequality

$$\mathcal{A}(\bar{y}, y-\bar{y}) \ge (y_d, y-\bar{y}) \qquad \forall\, y\in K. \tag{1.4}$$

According to the regularity results in [21,22,14], the solution $\bar{y}$ of (1.2) belongs to $C^2(\Omega)\cap H^3_{loc}(\Omega)$. On the other hand, since the state constraint is inactive near $\partial\Omega$ because $\psi > 0$ on $\partial\Omega$, the variational inequality (1.4) becomes an equality near $\partial\Omega$. Therefore the regularity of $\bar{y}$ near $\partial\Omega$ is determined by the elliptic regularity of the biharmonic equation with the boundary conditions of simply supported plates (cf. [4] and [13, Appendix A]). Hence $\bar{y}$ belongs to $H^{2+\alpha}$ in a neighborhood of $\partial\Omega$, where the elliptic regularity index $\alpha\in(0,2]$ is determined by the interior angles of $\Omega$. However $\bar{u} = -\Delta\bar{y}$ belongs to $H^1_0(\Omega)$ since the singularities of $\bar{y}$ at the corners of $\Omega$ are harmonic functions.

We can rewrite the variational inequality (1.4) in an equivalent form by using a Lagrange multiplier:

$$\mathcal{A}(\bar{y}, y) = (y_d, y) - \int_\Omega y\,d\bar{\lambda} \qquad \forall\, y\in H^2(\Omega)\cap H^1_0(\Omega), \tag{1.5a}$$

$$\int_\Omega (\psi - \bar{y})\,d\bar{\lambda} = 0, \tag{1.5b}$$

where $\bar{\lambda}$ is a nonnegative finite Borel measure.

The fact that the Lagrange multiplier $\bar{\lambda}$ is a measure and not a function in $L_2(\Omega)$ complicates the analysis of finite element methods for (1.2)–(1.4) considerably. Following the ideas introduced in [12,11] for the obstacle problem of clamped Kirchhoff plates, a quadratic $C^0$ interior penalty method for (1.2)–(1.4) was investigated in [13], where it was shown, without using (1.5), that the error for the approximation of the optimal state $\bar{y}$ in an $H^2$-like energy norm is $O(h^\alpha)$ on quasi-uniform meshes and $O(h)$ on properly graded meshes, where $h$ represents the mesh size.

In this paper we will compare three post-processing procedures that compute approximations of the optimal control $\bar{u} = -\Delta\bar{y}$ from the approximation of $\bar{y}$ obtained by the quadratic $C^0$ interior penalty method. The first procedure is based on numerical differentiation, the second procedure involves numerical differentiation and averaging, and the third procedure involves numerical differentiation and smoothing. We will demonstrate that even though the a priori $L_2$ error estimates for the approximate optimal controls generated by all three procedures are of the same magnitude as the error for the optimal state in the energy norm, in practice the second procedure performs better than the first one and the third procedure performs better than the second one. In particular the convergence of the approximate optimal control in the $H^1$ norm is observed for the third procedure even on domains with singular corners.

The rest of the paper is organized as follows. In Section 2 we review the quadratic $C^0$ interior penalty method and introduce the post-processing procedures. Since two of the procedures have been analyzed previously, we will focus on the analysis of the third procedure, which is carried out in Section 3. Numerical results that compare the performance of the post-processing procedures are reported in Section 4. We end the paper with some concluding remarks in Section 5.

Throughout the paper we will use $C$ to denote a generic positive constant independent of the mesh size $h$ that can take different values at different occurrences.

**Remark 1.1.** The reformulation of optimal control problems as fourth order variational inequalities is well-known in the literature (cf. for example [3,33,24,37]). The novelty of our approach is in the analysis, which shows that essentially all finite element methods that work for the boundary value problems of Kirchhoff plates also work for the variational inequalities, without assuming additional conditions on the free boundary. It would be interesting to find out whether our approach can be extended to optimal control problems with both state and control constraints that have been investigated in [34,28,16,30,39] by classical approaches.