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Comparison results for the Stokes equations

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ABSTRACT

This paper enfolds a medius analysis for the Stokes equations and compares different finite element methods (FEMs). A first result is a best approximation result for a P_1 non-conforming FEM. The main comparison result is that the error of the P_2P_0-FEM is a lower bound to the error of the Bernardi–Raugel (or reduced P_2P_0) FEM, which is a lower bound to the error of the P_1 non-conforming FEM, and this is a lower bound to the error of the MINI-FEM. The paper discusses the converse direction, as well as other methods such as the discontinuous Galerkin and pseudostress FEMs.
Furthermore this paper provides counterexamples for equivalent convergence when different pressure approximations are considered. The mathematical arguments are various conforming companions as well as the discrete inf-sup condition.

1. Introduction

Given some external force $f \in L^2(\Omega;\mathbb{R}^2)$ in some polygonal Lipschitz domain Ω, the Stokes equations seek the velocity field $u \in H^1_0(\Omega;\mathbb{R}^2) := \{u \in H^1(\Omega;\mathbb{R}^2) \mid u|_{\partial\Omega} = 0$ in the sense of traces$\}$ and the pressure distribution $p \in L^2_0(\Omega) := \{q \in L^2(\Omega) \mid \int_\Omega q\,dx = 0\}$ with

$$-\Delta u + \nabla p = f \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in } \Omega. \tag{1.1}$$

This paper compares several standard mixed finite element methods for the numerical approximation of the unknown solution pair $(u, p) \in H^1_0(\Omega;\mathbb{R}^2) \times L^2_0(\Omega)$ in terms of accuracy. Comparison results for the Poisson model problem of [7,12] give rise to the conjecture that first-order finite element methods (FEMs) for the Stokes problem are comparable in the sense that their errors on the same mesh are equivalent up to multiplicative constants, which are independent of the local mesh-size. The aim of this paper is to investigate the comparability of FEMs that are conceptually very different. The considered FEMs are MINI-FEM, CR-NCFEM, P_2P_0-FEM and BR-FEM (cf. Figs. 1–2). Since they use different continuous and discontinuous approximations of the velocity and/or the pressure, the approximation properties of the ansatz spaces do not allow for equivalence but only for a comparison in one direction.

The constraint $\operatorname{div} u = 0$ excludes standard piecewise affine FEMs based on continuous piecewise affine approximations of the velocity components (see, e.g., [8]). The MINI-FEM from Fig. 1(a) (see Section 2.3 for a precise definition) is a conforming

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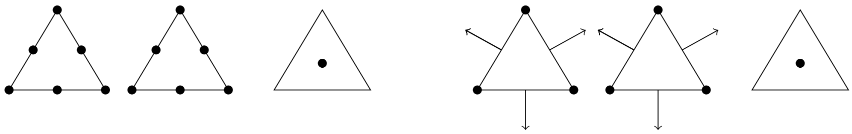
(a) MINI-FEM (b) CR-NCFEM

Fig. 1. MINI-FEM and CR-NCFEM for the Stokes equations.



(a) P_2P_0-FEM (b) BR-FEM

Fig. 2. P_2P_0-FEM and BR-FEM for the Stokes equations.

method which fulfils the constraint $\operatorname{div} u = 0$ in a weak sense only. It is based on a piecewise affine approximation of the velocity with an additional bubble function on each triangle for each component of the velocity.

The P_1 non-conforming FEM, CR-NCFEM, from Fig. 1(b) (see Section 2.3 for the precise definition), however, fulfils this constraint element-wise. While for the MINI-FEM the best approximation result

$$\|\nabla(u - u_{\mathrm{MINI}})\| + \|p - p_{\mathrm{MINI}}\| \lesssim \min_{v_{\mathrm{MINI}} \in V_{\mathrm{MINI}}(\mathcal{T})} \|\nabla(u - v_{\mathrm{MINI}})\| + \min_{q_{\mathrm{MINI}} \in P_1(\mathcal{T}) \cap C(\Omega) \cap L^2_0(\Omega)} \|p - q_{\mathrm{MINI}}\|$$

is a direct consequence of the conformity and stability, this paper proves the best approximation result

$$\|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\| + \|p - p_{\mathrm{CR}}\| \lesssim \min_{v_{\mathrm{CR}} \in V_{\mathrm{CR}}(\mathcal{T})} \|\nabla_{\mathrm{NC}}(u - v_{\mathrm{CR}})\| + \min_{q_{\mathrm{CR}} \in P_0(\mathcal{T}) \cap L^2_0(\Omega)} \|p - q_{\mathrm{CR}}\| + \operatorname{osc}(f, \mathcal{T})$$

for the CR-NCFEM. The notation $A \lesssim B$ abbreviates the inequality $A \leqslant CB$ with a mesh-size independent generic constant $C > 0$. The constant C may depend on the minimal angle in the triangulation but not on the local mesh-size. The best approximation result leads to the comparison

$$\|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\| + \|p - p_{\mathrm{CR}}\| \lesssim \|\nabla(u - u_{\mathrm{MINI}})\| + \|p - p_{\mathrm{MINI}}\| + \|h_{\mathcal{T}} f\|$$

with the additional term $\|h_{\mathcal{T}} f\|$ with the piecewise constant mesh-size $h_{\mathcal{T}}$.

The P_2P_0-FEM and the BR-FEM, from Fig. 2(a) and 2(b), approximate the velocity by piecewise P_2 and some enriched P_1 functions and the pressure by piecewise constant functions. The conformity of the P_2P_0-FEM and the inclusion $V_{\mathrm{BR}}(\mathcal{T}) \subseteq V_{\mathrm{P2}}(\mathcal{T})$ for the underlying finite element spaces of the velocity approximation of BR-FEM and P_2P_0-FEM imply

$$\|\nabla(u - u_{\mathrm{P2}})\| + \|p - p_{\mathrm{P2}}\| \lesssim \|\nabla(u - u_{\mathrm{BR}})\| + \|p - p_{\mathrm{BR}}\|.$$

Since there exist examples where the convergence of the P_2P_0-FEM is of second order and the BR-FEM is a first order method the converse direction of this estimate cannot be expected to hold in general (see Remark 4.5). The use of a conforming companion of the non-conforming solution $u_{\mathrm{CR}} \in V_{\mathrm{CR}}(\mathcal{T})$ of the CR-NCFEM yields

$$\|\nabla(u - u_{\mathrm{BR}})\| + \|p - p_{\mathrm{BR}}\| \lesssim \|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\| + \|p - p_{\mathrm{CR}}\|.$$

Altogether, the main comparison results of this paper read

$$\begin{aligned}\|\nabla(u - u_{\mathrm{P2}})\| + \|p - p_{\mathrm{P2}}\| &\lesssim \|\nabla(u - u_{\mathrm{BR}})\| + \|p - p_{\mathrm{BR}}\| \\ &\lesssim \|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\| + \|p - p_{\mathrm{CR}}\| \\ &\lesssim \|\nabla(u - u_{\mathrm{MINI}})\| + \|p - p_{\mathrm{MINI}}\| + \|h_{\mathcal{T}} f\|. \end{aligned} \tag{1.2}$$

Furthermore this paper discusses the pressure approximation by piecewise constant functions and by continuous piecewise affine functions. Theorem 4.9 proves that

$$\|p - p_h\| \lesssim \|\nabla(u - u_H)\| + \|p - p_H\| + \operatorname{osc}(f, \mathcal{T})$$

does *not* hold in general for solutions (u_h, p_h) and (u_H, p_H) of FEMs with piecewise constant resp. continuous piecewise affine approximations of the pressure. On the other hand, the continuity of the pressure approximation is not a natural restriction and causes that

$$\|p - p_H\| \lesssim \|\nabla_{\mathrm{NC}}(u - u_h)\| + \|p - p_h\|$$

does *not* hold in general.