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Parallel multilevel solvers for the cardiac electro-mechanical coupling

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ABSTRACT

We develop a parallel solver for the cardiac electro-mechanical coupling. The electric model consists of two non-linear parabolic partial differential equations (PDEs), the so-called Bidomain model, which describes the spread of the electric impulse in the heart muscle. The two PDEs are coupled with a non-linear elastic model, where the myocardium is considered as a nearly-incompressible transversely isotropic hyperelastic material. The discretization of the whole electro-mechanical model is performed by Q1 finite elements in space and a semi-implicit finite difference scheme in time. This approximation strategy yields at each time step the solution of a large scale ill-conditioned linear system deriving from the discretization of the Bidomain model and a non-linear system deriving from the discretization of the finite elasticity model. The parallel solver developed consists of solving the linear system with the Conjugate Gradient method, preconditioned by a Multilevel Schwarz preconditioner, and the non-linear system with a Newton–Krylov-Algebraic Multigrid solver. Three-dimensional parallel numerical tests on a Linux cluster show that the parallel solver proposed is scalable and robust with respect to the domain deformations induced by the cardiac contraction.

1. Introduction

We develop a parallel solver based on algebraic multigrid and multilevel Schwarz methods for the solution of the cardiac electro-mechanical coupling model. This model consists of the Bidomain equations (electrical model), a degenerate system of parabolic partial differential equations modeling the cardiac bioelectrical activity, coupled with a quasi-static mechanical model, describing the contraction and relaxation of the cardiac muscle during a heart beat.

The numerical approximation of the cardiac electro-mechanical coupling is a challenging multiphysics problem, because the space and time scales associated with the electrical and mechanical models are very different. The discretization of the model by finite elements in space and semi-implicit finite difference splitting methods in time yields at each time step the solution of a large ill-conditioned linear system, deriving from the discretization of the Bidomain equations, and of a non-linear system, deriving from the discretization of the non-linear elasticity equations.

Many studies have been devoted to the development of efficient solvers and preconditioners for the Bidomain model, see e.g. [8,13,30,32,44,31,40,54,41,42,58,62,63] and the surveys [38,60], but the robustness of these methods with respect to the domain deformation induced by the mechanical feedback has not been demonstrated yet. In the last years, several works

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have been devoted to the development of models for the mechanical cardiac contraction and to the numerical simulations of the electro-mechanical coupling models, see e.g. [21,24,12,23,36,37,49,34,55]. A few studies have focused on the development of efficient solvers for the quasi-static mechanical model, see [36,59] for a parallel GMRES solver and [25,24,23] for parallel direct solvers. The majority of cardiac mechanics simulation studies have used very coarse mechanical meshes in comparison with the standard electrical meshes. Thus, the solution of the quasi-static nonlinear mechanical model is performed solving the linear Jacobian system by direct methods, at each Newton iteration. However, in the recent paper [37], it has been shown that, when complex active tension development models are used, stability issues may arise in case of too coarse mechanical meshes, and fine meshes are needed also for the mechanical model.

The first aim of the present paper is to study the scalability and robustness, with respect to mechanically induced domain deformations, of Multilevel Schwarz methods for the Bidomain system. The second aim is to study the scalability of Algebraic Multigrid preconditioners for the linear Jacobian system arising at each Newton iteration during the solution of the non-linear elasticity equations in the mechanical model.

2. Mathematical models

2.1. Mechanical model

Let us denote the material coordinates of the undeformed or reference cardiac domain by $\mathbf{X} = (X_1, X_2, X_3)^T$, the spatial coordinates of the deformed cardiac domain by $\mathbf{x} = (x_1, x_2, x_3)^T$ and the region occupied by the undeformed and deformed, at time t, cardiac domains by $\hat{\Omega}$ and $\Omega(t)$, respectively. We denote by Div and div (Grad and grad) the material and spatial divergence (gradient) of a vector (scalar), respectively. From a mechanical point of view, the cardiac tissue is modeled as a non-linear elastic material. The deformation gradient tensor $\mathbf{F}$ and its determinant are given by

$$\mathbf{F}(\mathbf{X}, t) = \{F_{ij}\} = \left\{ \frac{\partial x_i}{\partial X_j} \quad i, j = 1, 2, 3 \right\}, \qquad J(\mathbf{X}, t) = \det \mathbf{F}(\mathbf{X}, t).$$

The Cauchy–Green deformation tensor $\mathbf{C}$ and Lagrange–Green strain tensor $\mathbf{E}$ are

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \quad \text{and} \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}),$$

where $\mathbf{I}$ denotes the identity matrix.

We first assume that the time-dependent inertial term in the governing elastic wave equation may be neglected, see e.g. [24,26,27,33,61,6]. Thus, the quasi-static Cauchy's equation of equilibrium, without body force, in term of the Cauchy stress tensor σ is given by $\operatorname{div} \sigma = \mathbf{0}$, in $\Omega(t)$ and in the coordinates of the deformed body satisfy the steady-state force equilibrium equation

$$\operatorname{Div}(\mathbf{FS}) = \mathbf{0}, \qquad \mathbf{X} \in \hat{\Omega}, \tag{1}$$

where $\mathbf{S} = \{s_{ij}\} = J\mathbf{F}^{-1}\sigma\mathbf{F}^{-T}$ is the second Piola–Kirchoff stress tensor. The tensor $\mathbf{S}$ is given by the sum of a passive elastic component $\mathbf{S}^{pas}$ and a biochemically generated active component $\mathbf{S}^{act}$, i.e. $\mathbf{S} = \mathbf{S}^{pas} + \mathbf{S}^{act}$, as done in many previous studies, see e.g. [22,59,27]. An alternative multiplicative strategy for combining the passive $\mathbf{S}^{pas}$ and active $\mathbf{S}^{act}$ components has been recently proposed in [7], see also [2,36,48].

The passive component $\mathbf{S}^{pas}$ is computed from a suitable strain energy function W and the Green–Lagrange strain $\mathbf{E}$

$$S_{ij}^{pas} = \frac{1}{2}\left(\frac{\partial W}{\partial E_{ij}} + \frac{\partial W}{\partial E_{ji}}\right) \quad i, j = 1, 2, 3. \tag{2}$$

A wide variety of strain energy functions W have been proposed and adopted in the literature, see e.g. [11,14,15,18,33,45,49,52,57]. We recall that the cardiac tissue consists of an arrangement of fibers that rotate counterclockwise from epi- to endocardium, and that have a laminar organization modeled as a set of muscle sheets running radially from epi- to endocardium, e.g. [28,57]. In this paper, we choose to model the myocardium as a transversely isotropic hyperelastic material, with the exponential strain energy function [59]

$$\begin{aligned} W &= \frac{1}{2}c\big(e^{Q} - 1\big), \\ Q &= b_{ll}E_{ll}^2 + b_{tn}\big(E_{nn}^2 + E_{tt}^2 + 2E_{nt}^2\big) + 2b_{lt}\big(E_{lt}^2 + E_{ln}^2\big), \end{aligned} \tag{3}$$

where the Lagrange–Green strain tensor is referred to the orthogonal local fiber coordinate system, consisting of the fiber direction (l), and two others orthogonal cross fiber directions. The material constant c scales the stress, b_{ll} and b_{tn} scale the material stiffness in the fiber and the two cross fiber directions, and b_{lt} scales the material rigidity under shear in the fiber-transverse plane.

The myocardium is modeled as nearly-incompressible material and, following [59], we add a bulk modulus K multiplying a volume change penalization term into the strain energy

$$W = \frac{1}{2}c\big(e^{Q} - 1\big) + K\big(\sqrt{\det(C)} - 1\big)^2. \tag{4}$$