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Semi-implicit finite volume level set method for advective motion of interfaces in normal direction

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In this paper a semi-implicit finite volume method is proposed to solve the applications with moving interfaces using the approach of level set methods. The level set advection equation with a given speed in normal direction is solved by this method. Moreover, the scheme is used for the numerical solution of eikonal equation to compute the signed distance function and for the linear advection equation to compute the so-called extension speed [1]. In both equations an extrapolation near the interface is used in our method to treat Dirichlet boundary conditions on implicitly given interfaces. No restrictive CFL stability condition is required by the semi-implicit method that is very convenient especially when using the extrapolation approach. In summary, we can apply the method for the numerical solution of level set advection equation with the initial condition given by the signed distance function and with the advection velocity in normal direction given by the extension speed. Several advantages of the proposed approach can be shown for chosen examples and application. The advected numerical level set function approximates well the property of remaining the signed distance function during whole simulation time. Sufficiently accurate numerical results can be obtained even with the time steps violating the CFL stability condition.

1. Introduction

The level set methods are a popular numerical tool to treat applications with moving interfaces [27,26]. The advantages of level set methods for these applications are known and well discussed in literature, see e.g. [28,30,27,1,6,26,20,25,15,7]. To track a $(d-1)$-dimensional dynamic interface, one can use a d-dimensional (possibly fixed and rectangular) grid that is used to solve related PDEs. The moving interface is then described implicitly as the zero level set of the so-called level set function obtained by the solution of a specific advection equation. The advection velocity in this equation is usually given in the normal direction to the evolving interface. For the level set methods, the advection speed in normal direction must be defined everywhere in the d-dimensional computational domain, and it has to coincide with the prescribed normal velocity at the moving interface.

There exist various high quality numerical methods for solving advection level set equations, see e.g. [27,26,25,12,22]. However, there are several important points which must be solved when treating practical applications.

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First of all, the initial level set function must be given appropriately. A typical choice for such function is a signed distance to the interface which is usually obtained by solving the eikonal equation [27,26] with zero Dirichlet boundary condition at the interface. The position of the interface coincides only rarely with the nodes of the d-dimensional grid, therefore a special treatment in grid points near the interface is necessary. Usually a kind of “brute force” method for assigning the distances near the interface is applied [27,1]. To avoid such brute force method we propose to use an extrapolation approach for the grid nodes near the interface. Such extrapolation technique is used for boundary conditions on implicitly given interfaces in several numerical methods like immersed interface methods [17,20], ghost fluid methods [6,14] or Cartesian grid methods [18,5,25].

The next important point is the choice of advection velocity for the level set method. The so-called “natural velocity” [1], or its normal component, is usually given in the whole computational domain, see e.g. our application in Section 4.3. In many applications the natural velocity is given only partially like on one side of the interface [21,7] or even on the interface only. Therefore some procedure to extend the natural velocity away from the position of interface to the whole domain must be at disposal in general.

In any case, the usage of inappropriate velocity in the advection equation can yield solutions with very steep and/or flat gradient in subregions of the computational domain which may deteriorate the accuracy of numerical solutions [28,1,16,15,10]. To avoid it, two main approaches are incorporated into the level set method. On the one hand, the so called reinitialization is used in order to describe the interface for whole simulation time with the signed distance function, see e.g. [28,16,10,15]. On the other hand, the so called “extension velocity” or “extension speed” [1] is constructed and used instead of the natural velocity to keep the evolving level set function (theoretically) equal to a signed distance during the whole interface evolution [30,1,7].

In our method we adopt the approach with the extension velocity. To compute it, an auxiliary linear advection equation with Dirichlet boundary condition (representing the natural speed) at the implicitly given interface is solved. Again, we prefer the extrapolation approach near the interface to treat such Dirichlet boundary conditions for the linear advection equation alternatively to some brute force method [30,1,2]. Consequently, both the (re-)initialization and the extension speed computation can be treated successfully by the extrapolation approach proposed in this paper.

The last but not least point which we discuss here is the precision and stability of the numerical solution obtained with the level set method. Clearly, high resolution methods without restrictive stability condition are desirable for practical applications. This is a general requirement when solving the advection equations, but even a must when treating the Dirichlet boundary conditions on implicitly given interface by an extrapolation technique.

When using the extrapolation with any explicit scheme having a natural CFL time step restriction (which is common for hyperbolic problems), one necessarily arrives to troubles, cf. the so called small cell problem in the Cartesian grid methods [18,5]. The reason is that the time step for the explicit scheme must be proportional to a shortest distance of the implicitly given interface to grid points which can be, in fact, arbitrary small. That makes standard explicit methods impractical for this type of problems and requires additional treatment inside the scheme [5,25,9,7]. On the other hand, the implicit (or semi-implicit) schemes can handle this phenomenon straightforwardly since they do not impose such restrictive conditions for coupling the temporal and spatial grid resolutions.

In this paper we use the second order semi-implicit finite volume method proposed in [22–24] and couple it with the extrapolation technique near the implicitly given interface. Moreover, we introduce to such scheme a different upwind type reconstruction of numerical solution on finite volume faces that is based on [12,13] which makes the method suitable for solving the stationary problems arising in the signed distance and extension speed computations. The proposed method is applied also to the advection level set equation for moving interface, therefore all three basic components of the level set method are treated in the same manner by using the proposed semi-implicit finite volume method.

The paper is organized as follows. In Section 2 we introduce partial differential equations (PDEs) used in the level set method and discretized by our semi-implicit finite volume scheme. In Section 3 we derive the proposed method for general advection equation and explain its usage in the case of moving interfaces. In Section 4 we present numerical experiments showing the properties of the proposed level set method.

2. PDEs for the level set method

Let $\Gamma(t) \subset R^d$ be a closed interface that evolves in time $t \in [0, T]$ in d-dimensional space. The movement of evolving interface $\Gamma(t)$ can be defined by describing the movement of all points located at the interface. More precisely, let $\Gamma(0)$ be a given initial position, then one requires

$$\Gamma(t) = \left\{X(t);\ X(0) \in \Gamma(0)\right\}, \quad t \in [0, T], \tag{1}$$

where a trajectory $X(t)$ describes the position of a particle at time t located initially at $X(0) \in \Gamma(0)$. We consider here only closed interfaces $\Gamma(t)$. By $\Omega(t) \subset R^d$ we denote a domain that is surrounded by the interface, i.e. $\partial\Omega(t) \equiv \Gamma(t)$.

We suppose that a velocity $\vec{V}$ is given in advance at any time $t \in [0, T]$ for each particle $\gamma \in \Gamma(0)$. Having such “natural” velocity $\vec{V}$ [1], the trajectories $X(t)$ can be determined by solving ordinary differential equations

$$\dot{X}(t) = \vec{V}\left(X(t), t\right), \quad t \in [0, T],\ X(0) \in \Gamma(0). \tag{2}$$